Zeros of Polynomials on Banach spaces: *The Real Story*

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Abstract

Let E be a real Banach space. We show that either E admits a positive definite 2-homogeneous polynomial or every 2-homogeneous polynomial on E has an infinite dimensional subspace on which it is identically zero. Under addition assumptions, we show that such subspaces are non-separable. We examine analogous results for nuclear and absolutely $(1, 2)$-summing 2-homogeneous polynomials and give necessary and sufficient conditions on a compact set K so that $C(K)$ admits a positive definite 2-homogeneous polynomial or a positive definite nuclear 2-homogeneous polynomial.

1 Introduction

The study of the zeros of a complex polynomial has a long history, with results coming via complex analysis, algebraic geometry, and functional analysis (see, e.g., [7], [8], [11]). On the other hand, similar studies for real polynomials are somewhat less common ([1]). The case of the polynomial $P : \mathbf{R}^n \to \mathbf{R},\ P(x) = \sum_{j=1}^n x_j^2$ notwithstanding, it is exactly the zeros of real valued $2-$homogeneous polynomials which will be of interest here, in the case when the domain $\mathbf{R}^n$ is replaced by an infinite dimensional real Banach space E. There are many 'large' Banach spaces E for which there is no positive definite $2-$homogeneous polynomial P. As we will see, for a real Banach space E, either E admits a positive definite $2-$homogeneous polynomial or every $2-$homogeneous polynomial on E is identically zero on an infinite dimensional subspace of E.

The plan of this article is as follows. In Section 2, we give several characterizations of positive definite polynomials. We prove several dichotomy results in Section 3 of the

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type indicated at the end of the preceding paragraph. A natural question of interest which arises is if, in the event that E does not admit a positive definite 2-homogeneous polynomial, what is the dimension of the subspace on which a $2-$homogeneous polynomial will vanish. This question will be only partially resolved here. Finally, in Section 4, we consider the special cases of $C(K)$ and absolutely (1,2)-summing polynomials, and we also present several examples and open questions.

We recall that an $n-$homogeneous polynomial $P : E \to \mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ is, by definition, the restriction to the diagonal of a necessarily unique symmetric continuous $n-$linear form $\check{P} : E \times ... \times E \to K$; that is, $P(x) = \check{P}(x, \ldots, x)$ for every $x \in E$. The polynomial P is said to be *positive definite* if $P(x) \geq 0$ for every x and $P(x) = 0$ implies that $x = 0$.

An n-homogeneous polynomial P on E is *nuclear* if there is bounded sequence $(\phi_j)_{j=1}^{\infty} \subset E'$ and a point $(\lambda_j)_{j=1}^{\infty}$ in ℓ_1 such that

$$P(x) = \sum_{j=1}^{\infty} \lambda_j \phi_j(x)^n$$

for every x in E. The space of all nuclear n-homogeneous polynomials on E is denoted by $\mathcal{P}_N(^nE)$. A sequence $(x_j)_j$ in E is said to be *weakly 2-summing* if

$$\sup_{\phi \in B_{E'}} \sum_{j=1}^{\infty} \phi(x_j)^2 < \infty.$$

An n-homogeneous polynomial P on E is said to be *absolutely (1,2)-summing* if P maps weakly 2-summing sequences into absolutely summing sequences; that is if $\sum_{j=1}^{\infty} \|P(x_j)\| < \infty$ for every weakly $2-$summing sequence $(x_j)_j$. It is shown in [10] (cf [6]) that P is (1,2)-summing if and only if there is $C > 0$ so that for every positive integer m and every $x_1, \ldots, x_m$ in E we have

$$\sum_{j=1}^{m} |P(x_j)| \leq C \left(\sup_{\phi \in B_{E'}} \sum_{j=1}^{m} \phi(x_j)^2 \right)^{\frac{n}{2}}.$$

For background on polynomials, the reader is referred to [4].

2 Characterizations of positive definite polynomials

In this section we will give a number of conditions which are equivalent to the existence of a positive definite $2-$homogeneous polynomial. A fundamental result which we will constantly use is the easily verified fact that if P is a $2-$homogeneous polynomial on the Banach space E, then P satisfies the parallelogram law: $P(x + y) + P(x - y) = 2(P(x) + P(y))$. Consequently, if P is also positive definite and we give E the norm $x \to \sqrt{P(x)}$, then E is a pre-Hilbert space with the associated symmetric bilinear form $\check{P}$ as inner product.

We begin with a very elementary necessary condition for the existence of positive definite $2-$homogeneous polynomials, in terms of the symmetric bilinear form $\check{P}$ associated to a $2-$homogeneous polynomial P.

Proposition 1 *A polynomial $P \in \mathcal{P}(^2E)$ is positive definite if and only if for every $x, y \in E$ such that $x \neq \pm y$,*

$$|\check{P}(x,y)| < 1/2(P(x) + P(y)).$$

Consequently, if P is a positive definite $2-$homogeneous polynomial on E, then $||P|| = ||\check{P}||$.

Proof. Assume that P is positive definite, and so $\check{P}$ is an inner product. Hence we may apply the Cauchy-Schwartz inequality: $|\check{P}(x,y)| \leq |P(x)P(y)|^{1/2}$, with equality if and only if $x = \pm y$. Next, by the arithmetic-geometric inequality, $|P(x)P(y)|^{1/2} \leq \frac{1}{2}(P(x) + P(y))$. The converse follows by taking an arbitrary $x \neq 0$ and $y = 0$ in the inequality.

Although parts of the following proposition may well be 'folklore,' we include a complete proof (cf [5]).

Proposition 2 *The following conditions on a Banach space E are equivalent:*

(i) E admits a positive definite $2-$homogeneous polynomial.

(ii) There is a continuous linear injection from E into a Hilbert space.

(iii) The point 0 is an exposed point of the convex cone of the subset $\{\delta_x \bigotimes \delta_x \ : \ x \in S_E\}$ of the symmetric tensor product $E \bigotimes_{\pi,s} E$, where S_E is the unit sphere of E.

(iv) There is a $2-$homogeneous polynomial P on E whose set of zeros is contained in a finite dimensional subspace of E.

Proof. (i) $\Rightarrow$ (ii): Let $\check{P}$ be the symmetric positive definite bilinear form associated to the positive definite polynomial P, so that $(E, \check{P})$ is a pre-Hilbert space with completion, say, H with the induced pre-Hilbert norm. Then the injection $j : E \to H$ is continuous since $\|j(x)\| = |\check{P}(x,x)|^{\frac{1}{2}} = |P(x)|^{\frac{1}{2}} \leq \|P\|^{\frac{1}{2}}\|x\|$.

(ii) $\Rightarrow$ (iii): Note that the space of $2-$homogeneous polynomials on E is the dual of $E\widehat{\bigotimes}_{\pi,s}E$. Also, recall that the convex cone of the set $\{\delta_x \bigotimes \delta_x : x \in S_E\}$ consists of all points of the form $\{\sum_{i=1}^n a_i\delta_{x_i} \bigotimes \delta_{x_i}$, where $x_i \in S_E$ and $a_i \geq 0\}$. Now, the polynomial $P(x) \equiv < j(x), j(x) >$ is positive definite on E. If we regard P as an element of $(E\widehat{\bigotimes}_{\pi,s}E)'$, we see that $P(0) = 0$ while $P(\delta_x \bigotimes \delta_x) = P(x) > 0$ for all $x \in S_E$. Consequently, for any point $\sum_{i=1}^n a_i\delta_{x_i} \bigotimes \delta_{x_i}$ in the convex cone, $P(\sum_{i=1}^n a_i\delta_{x_i} \bigotimes \delta_{x_i}) = \sum_{i=1}^n a_iP(x_i) \geq 0$, with equality if and only if all $a_i = 0$.

(iii) $\Rightarrow$ (iv): Let $T \in (E\widehat{\bigotimes}_{\pi,s}E)'$ be such that $T(0) = 0$ and $T(b) > 0$ for all b in the convex cone. In particular, for all $x \in S_E$, $P(x) \equiv T(\delta_x \bigotimes \delta_x) > 0$, so that $P^{-1}(0) = 0$.

(iv) $\Rightarrow$ (i): We only consider the non-trivial situation, when $\dim E = \infty$. Suppose that P is a $2-$homogeneous polynomial whose zero set is contained in the finite dimensional subspace V with basis, say, $\{v_1, ..., v_n\}$. We first observe that $P(x)$ is always positive or always negative, for all $x \in E\backslash V$. Otherwise, there would exist $x, y \in S_{E\backslash V}$ such that $P(x) < 0 < P(y)$. Let $\gamma : [0,1] \to E\backslash V$ be a curve linking x and y. Then $P \circ \gamma(t) = 0$ for some $t \in [0,1]$, which is a contradiction. So, without loss of generality, we assume that $P(x) \geq 0$ for all $x \in E$. Let $\Pi : E \to V$ be a projection, with $\Pi(x) = \sum_{i=1}^{n} a_i(\Pi(x))v_i$. Then, the $2-$homogeneous polynomial Q defined by $Q(x) \equiv P(x) + \sum_{i=1}^{n} a_i(\Pi(x))^2$ is positive definite. ■

Remark 3 *Suppose that there is a normalized sequence $(\phi_j)_j \in E'$ such that if $x \in E$, $\phi_j(x) = 0$ for all j, then $x = 0$. Then the mapping $x \in E \mapsto (\frac{1}{j}\phi_j(x))$ defines an injection into ℓ_2, and so Proposition 2 applies. In particular, any separable space and $C(K)$ spaces, when K is compact and separable, admit a positive definite $2-$homogeneous polynomial. On the other hand, $E = c_0(\Gamma)$ and $E = \ell_p(\Gamma)$, where Γ is an uncountable index set and $p > 2$, do not admit positive definite $2-$homogeneous polynomials.*

We also note that if there is a continuous linear injection $j : E \to \ell_2$, $j(x) = (j_n(x))$, then the mapping $x \mapsto (\frac{j_n(x)}{2^n})$ is a *nuclear* injection between these spaces. We have proved (ii) $\Rightarrow$ (iii) of the following separable version of Proposition 2.

Proposition 4 *Let E be a real Banach space. The following conditions are equivalent:*

(i) E admits a positive definite $2-$homogeneous nuclear polynomial.

(ii) E admits a continuous injection $j : E \to \ell_2$.

(iii) There is a nuclear injection $j : E \to \ell_2$ of the form $j(x) = \sum_{n=1}^{\infty} \chi_n(x)e_n$ with $(||\chi_n||) \in \ell_1$.

Proof. (i) $\Rightarrow$ (ii): If $P(x) = \sum_{n=1}^{\infty} \phi_n(x)^2$ is a positive definite nuclear polynomial on E, then $j(x) \equiv \sum_{n=1}^{\infty} \phi_n(x)e_n$ will satisfy (ii).

(iii) $\Rightarrow$ (i): Let $j : E \to \ell_2$ be a nuclear injection, $j(x) = \sum_{n=1}^{\infty} \chi_n(x)e_n$, where $(||\chi_n||)_n \in \ell_1$. Since $\bigcap_{n=1}^{\infty} \ker \chi_n = \{0\}$, it follows that the $2-$homogeneous polynomial $P : E \to \mathbf{R}$, $P(x) \equiv \sum_{n=1}^{\infty} \chi_n^2(x)$ is positive definite. Finally, P is nuclear since $\sum_{n=1}^{\infty} \|\chi_n\|^2 \leq (\sum_{n=1}^{\infty} \|\chi_n\|)^2 < \infty$. ■

3 Dichotomy results

As we observed in Remark 3, any 'small,' i.e. separable, Banach space admits a positive definite $2-$homogeneous polynomial. So, the following result is of interest only for non-separable spaces. As we will see in Section 4, we can improve our results if we restrict our attention to absolutely (1,2)-summing polynomials or to $C(K)$ spaces.

Roughly speaking, our goal in this and the next section will be to show that if E does not admit a positive definite $2-$homogeneous polynomial, then every $P \in P(^2E)$ vanishes on an infinite dimensional subspace of E. In certain situations, we will be able to discuss how 'large' this subspace is. Our general technique, however, is non-constructive, in that we will make repeated use of an argument by contradiction involving Zorn's Lemma. To avoid repetition, we now present a 'meta-argument' to which we will frequently appeal.

Theorem 5 *Let E be a real Banach space which does not admit a positive definite $2-$homogeneous polynomial. Then, for every $P \in P(^2E)$, there is an infinite dimensional subspace of E on which it is identically zero.*

Proof. Suppose E does not admit a positive definite $2-$homogeneous polynomial and that $P \in P(^2E)$. Let $\mathcal{S} = \{S\colon S \text{ is a subspace of } E \text{ and } P|_S \equiv 0\}$. Order $\mathcal{S}$ by inclusion and use Zorn's Lemma to deduce the existence of a maximal element S of $\mathcal{S}$. Suppose that S is finite dimensional. Let $v_1, \dots, v_n$ be a basis for S and let $T = \bigcap_{x \in S} \ker A_x = \bigcap_{i=1}^n \ker A_{v_i}$ where $A_x\colon E \to \mathbf{R}$ is the linear map which sends y in E to $\check{P}(x, y)$. We note that $S \subset T$. To see this suppose that $y \in S$. Then for every $s \in S$, $s + y$ is also in S. Since

$$0 = P(s + y) = P(s) + 2A_s(y) + P(y) = 2A_s(y)$$

for every $s \in S$ we see that $y \in T$.

Since S is finite dimensional we can write T as $T = S \bigoplus Y$ for some subspace Y of T. It is easy to see that all the zeros of $P|_T$ are contained in S. Therefore, either $P|_T$ or $-P|_T$ is positive definite on Y. Let us suppose, without loss of generality, that $P|_T$ is positive definite on Y. As S is n-dimensional we can find $\phi_1, \dots, \phi_n$ so that $P + \sum_{i=1}^n \phi_i^2$ is positive definite on T. Note that T has finite codimension in E and hence is complemented. Let π_T be the (continuous) projection of E onto T. Then $(P + \sum_{i=1}^n \phi_i^2) \circ \pi_T + \sum_{i=1}^n A_{v_i}^2$ is a positive definite polynomial on E, contradicting the fact that E does not admit such a polynomial. ■

Remark 6 *The argument given in the first paragraph of the above proof will be used in several places in the sequel. Note that the containment $S \subset T$ is independent of the fact that S is finite dimensional. This argument can also be applied to yield a non-constructive argument of the fact that every $\mathbf{C}-$valued polynomial P on an infinite dimensional complex Banach space E such that $P(0) = 0$ is identically 0 on an infinite dimensional subspace (cf [2], [11]).*

It is also worth noting that the arguments used to show the complex version of these results are purely algebraic, whereas continuity is needed in the real situation.

In light of Remark 3, the following result is not surprising.

Theorem 7 *Let E be a real Banach space of type 2. Then either E admits a positive definite 2-homogeneous polynomial* **or** *every $P \in P(^2E)$ has an non-separable subspace on which it is identically zero.*

Proof: Assume that E does not admit a positive definite $2-$homogeneous polynomial and let $P \in P(^2E)$. Let $S \subset E$ be a maximal subspace such that $P|_S \equiv 0$. If S is separable, the argument in the first paragraph of Theorem 5 shows that the subspace $T \subset E$ can be written $T = S \bigoplus_a Y$, where Y is an algebraic complement of S in T and where, without loss of generality, $P|_T$ is positive definite on Y. Then for every $s \in S$ and $t \in T$,

$$P(s+t) = P(s) + 2\check{P}_s(t) + P(t) = P(t) \geq 0.$$

Since S is separable, we can find a sequence $\{\phi_i\}_{i=1}^{\infty}$ in E' so that $\sum_{i=1}^{\infty} \phi_i^2$ is positive definite on S, and hence $P+\sum_{i=1}^{\infty} \phi_i^2$ is positive definite on T. Hence we have a continuous linear injection i of T into some Hilbert space $L_2(I)$. Since E is type 2, Maurey's Extension Theorem ([3], Theorem 12.22) allows us to extend i to a (not necessarily injective) linear map $\tilde{i}$ from E into $L_2(I)$. Finally, define a map j from E into $L_2(I) \bigoplus_{\ell_2} \ell_2$ by

$$j(x) = \left(\tilde{i}(x), \sum_{i=1}^{\infty} \frac{A_{v_i}(x)}{i^2 \|A_{v_i}\|} e_i \right),$$

where e_i is the i^{th} basis vector in ℓ_2. Since j is a continuous injection, E admits a positive definite polynomial, which is a contradiction. ■

Remark 8 *Although it is natural to conjecture that the conclusion of Theorem 7 above holds for any Banach space E, the authors have been unable to prove this. However, we can obtain strengthened results if we allow a different hypothesis on E :*

Theorem 9 *Let E be a real Banach space which does not admit a positive definite 4-homogeneous polynomial. Then for every $2-$homogeneous polynomial P on E, there is a non-separable subspace of E on which P is identically zero.*

In fact, we prove somewhat more in Theorem 10, below, which we will need later.

Theorem 10 *Let E be a real Banach space which does not admit a positive definite 4-homogeneous polynomial, and let $(\psi_k)_{k=1}^{\infty}$ be a sequence in E'. Then for any countable family $(P_j)_{j=1}^{\infty} \subset P(^2E)$, there is a non-separable subspace of $\bigcap_{k=1}^{\infty} \ker \psi_k$ on which each P_j is identically zero.*

Note that if E does not admit a positive definite $4-$homogeneous polynomial, then it cannot admit a positive definite $2-$homogeneous one either. An example of an E satisfying the hypotheses of Theorems 9 and 10 is $E = \ell_p(I)$, where I is an uncountable index set and $p > 4$.

Proof of Theorem 10: The argument begins in a similar way to our earlier proofs. As before, let S be a maximal element of $\mathcal{S} = \{S \colon S \text{ is a subspace of } \bigcap_{k=1}^{\infty} \ker \psi_k \text{ and } P_j|_S = 0 \text{ all } j\}$. Suppose that S is separable, with countable dense set $(v_i)_{i=1}^{\infty}$. Let $T = \bigcap_{k=1}^{\infty} \ker \psi_k \cap \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \ker (A_j)_{v_i}$. As before, $S \subset T$. We can write T as $T = S \bigoplus_a Y$

for some subspace Y of T. Since all the common zeros of $P_j|_T$, $j \in \mathbf{N}$, are contained in S, $\sum_{j=1}^{\infty} \frac{P_j^2}{j^2\|P_j\|^2}$ is positive definite on Y. As S is separable we can find $(\phi_i)_{i=1}^{\infty}$ so that $\sum_{j=1}^{\infty} \frac{P_j^2}{j^2\|P_j\|^2} + \sum_{i=1}^{\infty} \phi_i^4$ is positive definite on T. Then

$$\sum_{j=1}^{\infty} \frac{P_j^2}{j^2\|P_j\|^2} + \sum_{i=1}^{\infty} \phi_i^4 + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{(A_j)_{v_i}^4}{i^2 j^2 \|(A_j)_{v_i}\|^4} + \sum_{k=1}^{\infty} \frac{\psi_k^4}{k^2\|\psi_k\|^4}$$

is a positive definite polynomial on E, contradicting the fact that E does not admit such a polynomial. ■

Corollary 11 *Let E be a real Banach space which does not admit a positive definite* 4-*homogeneous polynomial. Then every $P \in P(^3E)$ is identically zero on a non-separable subspace of E.*

Proof: Consider $P \in P(^3E)$ and let S be a maximal element of $\mathcal{S} = \{S\colon S$ is a subspace of E and $P|_S \equiv 0\}$. Suppose that $(v_i)_{i=1}^{\infty}$ is a countable dense subset of S. Let $A_{v_i,v_j}\colon E \to \mathbf{R}$ be the linear map which sends x in E to $\check{P}(v_i, v_j, x)$ and $Q_{v_i}\colon E \to \mathbf{R}$ be the continuous 2-homogeneous polynomial which sends x in E to $\check{P}(v_i, x^2)$. By Theorem 10,

$$\bigcap_{i,j=1}^{\infty} \ker A_{v_i,v_j} \cap \bigcap_{i=1}^{\infty} \ker Q_{v_i}$$

contains a non-separable subspace which we denote by T. Suppose that $y \in T$ is such that $P(y) = 0$. Then for every $x = \sum_i \alpha_i v_i \in \operatorname{span} S$ and $\lambda \in \mathbf{R}$ we have

$$\begin{aligned} P(x + \lambda y) &= P(x) + 3\lambda A_{x,x}(y) + 3\lambda^2 \check{P}(x, y, y) + \lambda^3 P(y) \\ &= P(x) + 3\lambda \sum_{i,j} \alpha_i \alpha_j A_{v_i,v_j}(y) + 3\lambda^2 \sum_i \alpha_i Q_{v_i}(y) + \lambda^3 P(y) = 0. \end{aligned}$$

Hence, by continuity of P, $P(x + \lambda y) = 0$ for every $x \in S$. By maximality of S it follows that all the zeros of $P|_T$ are contained in S. Since S is separable, we can write T as $T = (S \cap T) \bigoplus_a Y$ for some non-separable subspace Y of T. Since all the zeros of $P|_T$ are contained in S, $P|_Y$ is a 3-homogeneous polynomial on an infinite dimensional space which has its only zero at the origin, an impossibility. ■

The final theorem in this section is a natural extension of the two preceding results.

Theorem 12 *Let E be a real Banach space which does not admit a positive definite homogeneous polynomial. Then, for every polynomial P on E such that* $P(0) = 0$, *there is a non-separable subspace of E on which P is identically zero.*

As in the case of Theorem 10, the proof of Theorem 12, which we omit, can be easily adapted to show that any countable family of polynomials (not necessarily homogeneous), having uniformly bounded degrees, on a real Banach space which does not admit a positive definite homogeneous polynomial will have a non-separable subspace on which they are all zero. An example to which Theorem 12 applies is $E = c_0(I)$, where I is an uncountable set.

4 Special cases: $C(K)$, nuclear and absolutely $(1,2)$-summing polynomials

In our context, there are four possible properties which a real Banach space E might have:
(1). There is no positive definite $2-$homogeneous polynomial on E. In this case, every $2-$homogeneous polynomial on E is zero on an infinite dimensional subspace. Examples of this were given in Remark 3.
(2). E admits a positive definite, non-absolutely (1,2)-summing, $2-$homogeneous polynomial on E, which corresponds to a non-$2-$summing injection of E into some Hilbert space. One example of this situation is when E is a non-separable Hilbert space.
(3). E admits a positive definite, absolutely (1,2)-summing, non-nuclear $2-$homogeneous polynomial on E. As we will show in Corollary 14, one instance of this occurs when E is a sufficiently large $L_\infty(\mu)$.
(4). E admits a positive definite nuclear $2-$homogeneous polynomial, in which case Proposition 4 applies. Examples include all separable spaces and ℓ_∞.

In this section, we restrict to cases (3) and (4) above. Not surprisingly, perhaps, we can prove much stronger results concerning the size of subspaces of zeros of polynomials. We begin with the following lemma, which shows the connection between positive definite *absolutely (1,2)-summing* polynomials and embeddings into Hilbert space and which should be compared with Propositions 2 and 4.

Lemma 13 *A real Banach space E admits a positive definite 2-homogeneous absolutely (1,2)-summing polynomial if and only if there is a continuous 2-summing injection from E into a Hilbert space.*

Proof: Given a 2-homogeneous polynomial P on E we define a semi-norm $\|\,.\,\|_H$ on E by $\|x\|_H = \sqrt{|P(x)|}$. We know that E admits a positive definite 2-homogeneous polynomial P if and only if H, the completion of $(E, \|\,.\,\|_H)$, is a Hilbert space. When this occurs, $j\colon E \to H$, $j(x) = x$, is a continuous injection. The map j is 2-summing if and only if there is $C > 0$ such that for any finite sequence of vectors, $(x_i)_{i=1}^m$, in E we have

$$\sum_{i=1}^m \|j(x_i)\|_H^2 \le C \sup_{\phi\in B_{E'}} \sum_{i=1}^m \phi(x_i)^2.$$

This is equivalent to saying that there is $C > 0$ such that for any finite sequence of vectors, $(x_i)_{i=1}^m$, in E we have

$$\sum_{i=1}^m P(x_i) \le \ C \sup_{\phi\in B_{E'}} \sum_{i=1}^m \phi(x_i)^2,$$

which occurs if and only if P is an absolutely (1,2)-summing polynomial. ■

Corollary 14 *Let E be an $\mathcal{L}_{\infty,\lambda}$-space for some real λ ([3]). Then every positive definite polynomial on E is absolutely (1,2)-summing.*

Proof: By Theorem 3.7 of [3] every linear map from E into a Hilbert space is 2-summing. ■

Note, though, that there may well not exist any positive definite polynomials on an $\mathcal{L}_{\infty,\lambda}$ space.

We next consider the question of the existence of positive definite 2−homogeneous polynomials in case E is a $C(K)$ space. We recall that a (Borel) measure μ on a compact set K is said to be *strictly positive* if $\mu(B) > 0$ for every non-empty open set $B \subset K$.

Corollary 15 *Let $E = C(K)$ where K is a compact Hausdorff space. Then*

(i) $C(K)$ admits a positive definite 2-homogeneous polynomial if and only if K admits a strictly positive measure.

(ii) $C(K)$ admits a positive definite 2-homogeneous nuclear polynomial if and only if there is a sequence of finite Borel measures $(\mu_n)_{n=1}^{\infty}$ on K such that $\int_K f(x)\, d\mu_n(x) = 0$ for all n implies $f \equiv 0$.

Proof: (i). By Proposition 2, $C(K)$ admits a positive definite 2-homogeneous polynomial if and only if there is a continuous injection j from $C(K)$ to a Hilbert space. By Theorem 3.5 of [3] any such injection must be 2-summing and hence by an application of the Pietsch Factorization Theorem (see, e.g., Corollary 2.15 of [3]), the mapping j factors through the canonical inclusion j_2 of $C(K)$ into $L_2(\mu)$ for some finite regular Borel measure μ on K. Since j is injective, j_2 is injective. It is easily seen that j_2 being injective is equivalent to μ being a positive definite measure (see, e.g., [3], p. 42). The converse is straightforward, from Proposition 2.

(ii). As we observed in Proposition 4, a Banach space E admits a positive definite nuclear polynomial if and only if there is a sequence $\{\phi_n\}_{n=1}^{\infty}$ in E' with the property that $\bigcap_{n=1}^{\infty} \ker \phi_n = \{0\}$. As the dual of a $C(K)$ space is the set of all regular finite Borel measures on K the result follows. ■

Lemma 13 also allows us to prove nuclear and integral polynomial versions of Theorems 5 and 7.

Theorem 16 *Let E be a real Banach space.*

(i) Either E admits a positive definite 2-homogeneous nuclear polynomial **or** *every $P \in P_N(^2E)$ has a non-separable subspace on which it is identically zero.*

(ii) Either E admits a positive definite 2-homogeneous absolutely (1,2)-summing polynomial **or** *every absolutely (1,2)-summing has an non-separable subspace on which it is identically zero.*

Proof: (i) We reason as before, supposing that E does not admit a positive 2-homogeneous nuclear polynomial and that $P \in P_N(^2E)$. Let S be a maximal subspace of E on which P is identically 0, assume that $S = \overline{\{v_i \ : \ i \in \mathbb{N}\}}$, let $T = \cap_{i=1}^{\infty} \ker A_{v_i}$, and write $T = S \bigoplus_a Y$. Without loss of generality, we may assume that $P|_T$ is positive definite on Y, so that $P|_T \geq 0$.

Since S is separable, we can find a sequence $\{\phi_i\}_{i=1}^{\infty}$ in E' so that $\sum_{i=1}^{\infty} \phi_i^2$ is positive definite on S and nuclear on E. Hence $P + \sum_{i=1}^{\infty} \phi_i^2$ is positive definite and nuclear on T. We therefore have a continuous linear nuclear injection i of T into ℓ_2. We can extend i to a nuclear linear map $\tilde{i}$ from E into ℓ_2 (see, e.g., [9]).

Define a map $j : E \to \ell_2 \bigoplus_2 \ell_2$ by

$$j(x) = \left(\tilde{i}(x), \sum_{i=1}^{\infty} \frac{A_{v_i}(x)}{i^2 \|A_{v_i}\|} e_i \right).$$

Since j is a nuclear injection, E admits a positive definite nuclear polynomial, which is a contradiction.

(ii) The argument given above works in the absolutely (1,2)-summing case, the only significant change being an appeal to the Π_2 Extension Theorem (Theorem 4.15, [3]) to prove the existence of a $2-$summing extension mapping $\tilde{i} : E \to L_2(I) \bigoplus_2 \ell_2$, for a sufficiently large index set I. ∎

Even if we know that an $\mathcal{L}_{\infty,\lambda}$-space admits a positive definite absolutely (1,2)-summing polynomial, it is nevertheless possible to conclude something about the zeros of those $2-$homogeneous polynomials which are not absolutely (1,2)-summing.

Theorem 17 *Let E be a real $\mathcal{L}_{\infty,\lambda}$-space. Then every $P \in P(^2E)$ which is not absolutely (1,2)-summing has an infinite dimensional subspace on which it is identically zero.*

Proof: Suppose $P \in P(^2E)$ is not absolutely (1,2)-summing. Suppose that a maximal subspace S on which P vanishes is only finite dimensional, with basis $\{v_1, \ldots, v_n\}$. Let $T = \bigcap_{i=1}^{n} \ker A_{v_i}$, and write $T = S \bigoplus Y$, for some complemented subspace $Y \subset T$. Without loss of generality, $P|_T$ is positive definite on Y and, since S is finite dimensional, we can find $\phi_1, \ldots, \phi_n$ so that $P + \sum_{i=1}^{n} \phi_i^2$ is positive definite on T. Let π_T be the (continuous) projection of E onto T. Then $(P + \sum_{i=1}^{n} \phi_i^2) \circ \pi_T + \sum_{i=1}^{n} A_{v_i}^2$ is positive definite on E. But E is an $\mathcal{L}_{\infty,\lambda}$-space and so by Corollary 14, $(P + \sum_{i=1}^{n} \phi_i^2) \circ \pi_T + \sum_{i=1}^{n} A_{v_i}^2$ is absolutely (1,2)-summing implying that $P|_T$ and hence P itself is absolutely (1,2)-summing, a contradiction. ∎

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