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The Bishop–Phelps–Bollobás theorem for $\mathcal{L}(L_1(\mu), L_\infty[0,1])$

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Abstract

We show that the Bishop–Phelps–Bollobás theorem holds for all bounded operators from $L_1(\mu)$ into $L_\infty[0,1]$, where μ is a σ-finite measure.

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1. Introduction

In 1961, Bishop and Phelps [5] proved the celebrated Bishop–Phelps theorem, which shows that for every Banach space X, every element in its dual space X^* can be approximated by ones that attain their norms. Since then, this theorem has been extended to linear operators between Banach spaces [7,11,13,14,16], and also to nonlinear mappings [1,4,2,8,12]. On the other hand, Bollobás [6] sharpened it to apply a problem about the numerical range of an operator, now known as Bishop–Phelps–Bollobás theorem. We denote the unit sphere of a Banach space X by S_X, the closed unit ball by B_X, as usual.

Theorem 1.1 *(Bishop–Phelps–Bollobás theorem). Suppose $x \in S_X$, $f \in S_{X^*}$ and $|f(x) - 1| \leqslant \epsilon^2/2$ $(0 < \epsilon < \frac{1}{2})$. Then there exist $y \in S_X$ and $g \in S_{X^*}$ such that $g(y) = 1$, $\|f - g\| < \epsilon$ and $\|x - y\| < \epsilon + \epsilon^2$.*

Recently, Acosta, Aron, García and Maestre [3] defined the Bishop–Phelps–Bollobás property for a pair of Banach spaces. A pair of Banach spaces (X, Y) is said to have the Bishop–Phelps–Bollobás property for operators (*BPBP*) if for every $\epsilon > 0$ there are $\eta(\epsilon) > 0$ and $\beta(\epsilon) > 0$ with $\lim_{\epsilon \to 0} \beta(\epsilon) = 0$ such that for all $T \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ satisfying $\|T(x_0)\| > 1 - \eta(\epsilon)$, there exist a point $u_0 \in S_X$ and an operator $S \in S_{\mathcal{L}(X,Y)}$ that satisfy the following conditions:

$$\|Su_0\| = 1, \qquad \|u_0 - x_0\| < \beta(\epsilon), \quad \text{and} \quad \|S - T\| < \epsilon.$$

This property is a uniform one in nature.

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and (I, Σ, m) be the Lebesgue measure space, where $I = [0, 1]$. Finet and Payá [10] showed that the set of all norm attaining operators is dense in the space $\mathcal{L}(L_1(\mu), L_\infty(m))$. Further, we will show in this paper that the pair $(L_1(\mu), L_\infty(m))$ has the *BPBP*.

2. The result

It is well known that the space $\mathcal{L}(L_1(\mu), L_\infty(m))$ is isometrically isomorphic to the space $L_\infty(\mu \otimes m)$, where $\mu \otimes m$ denotes the product measure on $\Omega \times I$. More precisely, the operator $\hat{h}$ corresponding to an essentially bounded function h is given by

$$\big[\hat{h}(f)\big](t) = \int_\Omega h(\omega, t) f(\omega)\, d\mu(\omega)$$

for m-almost every $t \in I$ and for all $f \in L_1(\mu)$ (see [9]).

We recall the Lebesgue density theorem: *given a measurable set $E \subset \mathbb{R}$, we have $m(E \Delta \delta(E)) = 0$, where $\delta(E)$ is the set of points $y \in \mathbb{R}$ of density of E, that is,*

$$\delta(E) = \left\{ y \in \mathbb{R}\colon \lim_{h \to 0} \frac{m(E \cap [y - h, y + h])}{2h} = 1 \right\},$$

and $E \Delta \delta(E)$ is the symmetric difference of the sets E and $\delta(E)$. In addition, the closed unit ball of $L_1(m)$ is the closed absolutely convex hull of the set $\{\frac{\chi_B}{m(B)}\colon B \in \Sigma,\ 0 < m(B) < \infty\}$, equivalently,

$$\|g\|_\infty = \sup\left\{ \frac{1}{m(B)} \left| \int_B g\, dm \right| : B \in \Sigma,\ 0 < m(B) < \infty \right\}$$

for every $g \in L_\infty(m)$. For a measurable subset M of $\Omega \times I$, let $M_x = \{y \in I: (x, y) \in M\}$ for each $x \in \Omega$ and $M^y = \{x \in \Omega: (x, y) \in M\}$ for each $y \in I$.

Lemma 2.1. *Let M be a measurable subset of $\Omega \times I$ with positive measure, $0 < \epsilon < 1$, and $f_0 = \sum_{j=1}^m \alpha_j \frac{\chi_{A_j}}{\mu(A_j)} \in S_{L_1(\mu)}$, where each A_j is a measurable subset of Ω with finite positive measure, $A_k \cap A_l = \emptyset$, $k \neq l$, and α_j is a positive real number for every $j = 1, \ldots, m$ with $\sum_{j=1}^m \alpha_j = 1$. If $\|\hat{\chi}_M(f_0)\|_\infty > 1 - \epsilon$, then there exists a simple function $g_0 \in S_{L_1(\mu)}$ such that*

$$\left\| (\hat{\chi}_M + \hat{\varphi})(g_0) \right\|_\infty = 1 \quad \text{and} \quad \|f_0 - g_0\|_1 < \frac{4\epsilon}{1-\epsilon},$$

for any simple function φ in $L_\infty(\mu \otimes m)$ such that $\|\varphi\|_\infty \leqslant 1$ and φ vanishes on M.

Proof. Since $\|\hat{\chi}_M(f_0)\|_\infty > 1 - \epsilon$, there is a measurable subset B of I such that $0 < m(B)$ and

$$\left| \left\langle \hat{\chi}_M(f_0), \frac{\chi_B}{m(B)} \right\rangle \right| > 1 - \epsilon.$$

For each $j = 1, \ldots, m$ we put $M_j = M \cap (A_j \times B)$ and let

$$H_j = \left\{ (x, y): x \in A_j,\ y \in \delta\big((M_j)_x\big) \right\}.$$

As in the proof of Proposition 5 in [15], H_j's are disjoint measurable subsets of $\Omega \times I$ and $(\mu \otimes m)(H) > 0$, where $H = \bigcup_{j=1}^m H_j$. Then there is $y \in I$ such that $\mu(H^y) > 0$. We also note that for each $j = 1, \ldots, m$ we have $H_j \subset A_j \times \delta(B)$ and $(\mu \otimes m)(M_j \Delta H_j) = 0$. Let

$$J(y) = \left\{ j: \mu\big(H_j^y\big) > 0,\ 1 \leqslant j \leqslant m \right\}.$$

For $y \in \delta(B)$ with $J(y) \neq \emptyset$ we define $g_y \in S_{L_1(\mu)}$ by

$$g_y = \sum_{j \in J(y)} \beta_j \frac{\chi_{H_j^y}}{\mu(H_j^y)},$$

where $\beta_j = \alpha_j / (\sum_{k \in J(y)} \alpha_k)$.

We first claim that $\hat{\chi}_M + \hat{\varphi}$ attains its norm at g_y for every y with $\mu(H^y) > 0$.

Fix such y and let $B_n = [y - \gamma_n, y + \gamma_n]$, where (γ_n) is a sequence of positive numbers converging to 0. Note that for every $x \in H_j^y$ we have $(x, y) \in H_j$, which implies that

$$\lim_{n \to \infty} \frac{m((M_j)_x \cap B_n)}{m(B_n)} = 1.$$

The Lebesgue dominated convergence and Fubini theorems show that for each $j \in J(y)$

$$1=\lim_{n\to\infty}\frac{1}{\mu(H_j^y)}\int\limits_{H_j^y}\frac{m((M_j)_x\cap B_n)}{m(B_n)}\,d\mu(x)=\lim_{n\to\infty}\frac{(\mu\otimes m)(M_j\cap(H_j^y\times B_n))}{\mu(H_j^y)m(B_n)}.$$

On the other hand, since the simple function φ is assumed to vanish on M and also $\|\varphi\|_\infty\leqslant 1$, we have

$$\begin{aligned}\left|\left\langle\hat{\varphi}\left(\frac{\chi_{H_j^y}}{\mu(H_j^y)}\right),\frac{\chi_{B_n}}{m(B_n)}\right\rangle\right|&=\left|\frac{1}{\mu(H_j^y)m(B_n)}\int\limits_{H_j^y\times B_n}\varphi\,d(\mu\otimes m)\right|\\&\leqslant\frac{(\mu\otimes m)((H_j^y\times B_n)\setminus M_j)}{\mu(H_j^y)m(B_n)}\\&=1-\frac{(\mu\otimes m)(M_j\cap(H_j^y\times B_n))}{\mu(H_j^y)m(B_n)}\to 0,\end{aligned}$$

as $n\to\infty$.

Therefore,

$$\begin{aligned}1\geqslant\big\|(\hat{\chi}_M+\hat{\varphi})(g_y)\big\|_\infty&\geqslant\lim_{n\to\infty}\left|\left\langle(\hat{\chi}_M+\hat{\varphi})\left(\sum_{j\in J(y)}\beta_j\frac{\chi_{H_j^y}}{\mu(H_j^y)}\right),\frac{\chi_{B_n}}{m(B_n)}\right\rangle\right|\\&\geqslant\lim_{n\to\infty}\sum_{j\in J(y)}\beta_j\frac{(\mu\otimes m)(M\cap(H_j^y\times B_n))}{\mu(H_j^y)m(B_n)}\\&\quad-\lim_{n\to\infty}\sum_{j\in J(y)}\beta_j\left|\frac{1}{\mu(H_j^y)m(B_n)}\int\limits_{H_j^y\times B_n}\varphi\,d(\mu\otimes m)\right|\\&\geqslant\lim_{n\to\infty}\sum_{j\in J(y)}\beta_j\frac{(\mu\otimes m)(M_j\cap(H_j^y\times B_n))}{\mu(H_j^y)m(B_n)}\\&\quad-\lim_{n\to\infty}\sum_{j\in J(y)}\beta_j\left[1-\frac{(\mu\otimes m)(M_j\cap(H_j^y\times B_n))}{\mu(H_j^y)m(B_n)}\right]=1,\end{aligned}$$

which shows that $\hat{\chi}_M+\hat{\varphi}$ attains its norm at g_y.

Next we claim that there exists $y\in\delta(B)$ such that $\mu(H^y)>0$ and

$$\|g_y-f_0\|_1<\frac{4\epsilon}{1-\epsilon}.$$

For each $j=1,\ldots,m$ we set $B_j^+=\{y\in\delta(B)\colon\ \mu(H_j^y)>0\}$, $B_j^0=\{y\in\delta(B)\colon\ \mu(H_j^y)=0\}$ and $B^0=\bigcap_{j=1}^m B_j^0$. By applying Fubini's theorem the sets B_j^+ and B_j^0 are Lebesgue measurable subsets of $[0,1]$.

We note that for each $j = 1, \dots, m$

$$(\mu \otimes m)(M_j) = (\mu \otimes m)\big(\big(A_j \times \delta(B)\big) \cap H_j\big)$$
$$= (\mu \otimes m)\big(\big(A_j \times \delta(B)\big) \cap \big\{(x, y) \in H_j\colon\ \mu\big(H_j^y\big) > 0\big\}\big).$$

Since

$$\left|\hat{\chi}_M(f_0)\left(\frac{\chi_B}{m(B)}\right)\right| > 1 - \epsilon,$$

we have

$$1 - \epsilon < \sum_{j=1}^{m} \alpha_j \frac{(\mu \otimes m)(M_j)}{(\mu \otimes m)(A_j \times B)},$$

which implies that

$$\sum_{j=1}^{m} \alpha_j \frac{(\mu \otimes m)((A_j \times \delta(B)) \setminus \{(x, y) \in H_j\colon\ \mu(H_j^y) > 0\})}{(\mu \otimes m)(A_j \times B)} < \epsilon, \tag{1}$$

and

$$\sum_{j=1}^{m} \alpha_j \frac{(\mu \otimes m)((A_j \times B_j^0))}{(\mu \otimes m)(A_j \times B)}$$
$$\leqslant \sum_{j=1}^{m} \alpha_j \frac{(\mu \otimes m)((A_j \times \delta(B)) \setminus \{(x, y) \in H_j\colon\ \mu(H_j^y) > 0\})}{(\mu \otimes m)(A_j \times B)} < \epsilon, \tag{2}$$

which implies that

$$\sum_{j=1}^{m} \alpha_j m\big(B_j^0\big) < \epsilon m(B). \tag{3}$$

It follows from this inequality that $m(B^0) < \epsilon m(B)$. For $y \in \delta(B) \setminus B^0$,

$$\|g_y - f_0\|_1 = \sum_{j \notin J(y)} \alpha_j + \sum_{j \in J(y)} \left[\left(\frac{\beta_j}{\mu(H_j^y)} - \frac{\alpha_j}{\mu(A_j)}\right)\mu\big(H_j^y\big) + \alpha_j \frac{\mu(A_j \setminus H_j^y)}{\mu(A_j)}\right]$$
$$= \sum_{j \notin J(y)} \alpha_j + 1 + \sum_{j \in J(y)} \left[-\alpha_j \frac{\mu(H_j^y)}{\mu(A_j)} + \alpha_j \frac{\mu(A_j \setminus H_j^y)}{\mu(A_j)}\right]$$
$$= 2 \sum_{j \notin J(y)} \alpha_j + \sum_{j \in J(y)} 2\alpha_j \frac{\mu(A_j \setminus H_j^y)}{\mu(A_j)}.$$

Assume that there is no $y \in \delta(B) \setminus B^0$ such that

$$\|g_y - f_0\|_1 < \frac{4\epsilon}{1-\epsilon}.$$

Then

$$\begin{aligned} \frac{4\epsilon}{1-\epsilon} m\big(\delta(B) \setminus B^0\big) &\leqslant \int\limits_{\delta(B)\setminus B^0} \|g_y - f_0\|_1 \, dm(y) \\ &= 2 \int\limits_{\delta(B)\setminus B^0} \left(\sum_{j \notin J(y)} \alpha_j + \sum_{j \in J(y)} \alpha_j \frac{\mu(A_j \setminus H_j^y)}{\mu(A_j)} \right) dm(y). \end{aligned}$$

It follows from the inequalities (1)–(3) that

$$\begin{aligned} \int\limits_{\delta(B)\setminus B^0} \sum_{j \notin J(y)} \alpha_j \, dm(y) &= \int\limits_{\delta(B)\setminus B^0} \sum_{J=1}^{m} \big(\alpha_j \chi_{B_j^0}(y)\big) \, dm(y) \\ &\leqslant \sum_{j=1}^{m} \alpha_j m\big(B_j^0\big) < \epsilon m(B), \end{aligned}$$

and

$$\begin{aligned} &\int\limits_{\delta(B)\setminus B^0} \sum_{j \in J(y)} \alpha_j \frac{\mu(A_j \setminus H_j^y)}{\mu(A_j)} \, dm(y) \\ &\quad = \int\limits_{\delta(B)\setminus B^0} \sum_{j=1}^{m} \left(\alpha_j \frac{\mu(A_j \setminus H_j^y)}{\mu(A_j)} \chi_{B_j^+}(y) \right) dm(y) \\ &\quad = \sum_{j=1}^{m} \alpha_j \frac{(\mu \otimes m)((A_j \times B_j^+) \setminus \{(x,y) \in H_j^y \colon y \in B_j^y\})}{\mu(A_j)} < \epsilon \, m(B). \end{aligned}$$

Therefore,

$$\begin{aligned} 4\epsilon m(B) &< \frac{4\epsilon}{1-\epsilon} m\big(\delta(B) \setminus B^0\big) \\ &\leqslant \int\limits_{\delta(B)\setminus B^0} \|g_y - f_0\|_1 \, dm(y) < 4\epsilon m(B), \end{aligned}$$

which is a contradiction. $\square$

Lemma 2.2. *(See [3, Lemma 3.3].) Let $\{c_n\}$ be a sequence of complex numbers with $|c_n| \leqslant 1$ for every n, and let $\eta > 0$ be such that for a convex series $\sum_{n=1}^{\infty} \alpha_n$, $\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta$. Then for every $0 < r < 1$, the set $A = \{i \in \mathbb{N}\colon \operatorname{Re} c_i > r\}$ satisfies the estimate*

$$\sum_{i \in A} \alpha_i \geqslant 1 - \frac{\eta}{1 - r}.$$

We recall that the set of simple functions is a dense subspace of $L_\infty(\mu \otimes m)$.

Theorem 2.3. *For the complex Banach spaces $L_1(\mu)$ and $L_\infty(m)$, let $T : L_1(\mu) \to L_\infty(m)$ be a bounded operator such that $\|T\| = 1$. Given $0 < \epsilon < 1/5$ and $f_0 \in S_{L_1(\mu)}$ satisfying $\|T(f_0)\|_\infty > 1 - \epsilon^8$, there exist $S \in \mathcal{L}(L_1(\mu), L_\infty(m))$, $\|S\| = 1$ and $g_0 \in S_{L_1(\mu)}$ such that*

$$\big\|S(g_0)\big\|_\infty = 1, \qquad \|T - S\| < \epsilon \quad \text{and} \quad \|f_0 - g_0\|_1 < 2\epsilon^4 + \frac{4\epsilon}{1 - \epsilon}.$$

Proof. Since the set of all simple functions is dense in $L_1(\mu)$, we may assume

$$f_0 = \sum_{j=1}^{m} \alpha_j \frac{\chi_{A_j}}{\mu(A_j)} \in S_{L_1(\mu)},$$

where each A_j is a measurable subset of Ω with finite positive measure, $A_k \cap A_l = \emptyset$, $k \neq l$, and every α_j is a nonzero complex number with $\sum_{j=1}^{m} |\alpha_j| = 1$. We may also assume that $0 < \alpha_j \leqslant 1$ for every $j = 1, \ldots, m$. Indeed, define $\Psi : L_1(\mu) \to L_1(\mu)$ by

$$\Psi(f) = \sum_{j=1}^{m} e^{-i\theta_j} f \cdot \chi_{A_j} + f \cdot \chi_{(\Omega \setminus \bigcup_{j=1}^{m} A_j)},$$

where $\theta_j = \arg(\alpha_j)$ for every $j = 1, \ldots, m$. The operator Ψ is an isometric isomorphism of $L_1(\mu)$ onto $L_1(\mu)$,

$$\big\|T(f_0)\big\|_\infty = \big\|\big(T \circ \Psi^{-1}\big)\big(\Psi(f_0)\big)\big\|_\infty > 1 - \epsilon^8$$

and

$$\Psi(f_0) = \sum_{j=1}^{m} |\alpha_j| \frac{\chi_{A_j}}{\mu(A_j)},$$

hence we may replace T and f_0 by $T \circ \Psi^{-1}$ and $\Psi(f_0)$, respectively.

Let h be the element in $L_\infty(\Omega \times I, \mu \otimes m)$, $\|h\|_\infty = 1$ corresponding to T, that is, $T = \hat{h}$. We can find a simple function

$$h_0 \in L_\infty(\Omega \times I, \mu \otimes m), \quad \|h_0\|_\infty = 1$$

such that $\|h - h_0\|_\infty < \|T(f_0)\|_\infty - (1-\epsilon^8)$, hence $\|\hat{h}_0(f_0)\|_\infty > 1-\epsilon^8$. We can write $h_0 = \sum_{l=1}^{p} c_l \chi_{D_l}$, where each D_l is a measurable subset of $\Omega \times I$ with positive measure, $D_k \cap D_l = \emptyset$, $k \neq l$, the complex number $|c_l| \leqslant 1$ for every $l = 1, \ldots, p$, and $|c_{l_0}| = 1$ for some $1 \leqslant l_0 \leqslant p$.

Let B be a Lebesgue measurable subset of I with $0 < m(B) < \infty$ such that

$$\left|\left\langle \hat{h}_0(f_0), \frac{\chi_B}{m(B)} \right\rangle\right| > 1 - \epsilon^8.$$

Choose $\theta \in \mathbb{R}$ so that

$$\begin{aligned} 1-\epsilon^8 &< \left|\left\langle \hat{h}_0(f_0), \frac{\chi_B}{m(B)} \right\rangle\right| \\ &= e^{i\theta}\left\langle \hat{h}_0(f_0), \frac{\chi_B}{m(B)} \right\rangle \\ &= \sum_{j=1}^{m} \alpha_j e^{i\theta}\left\langle \hat{h}_0\left(\frac{\chi_{A_j}}{\mu(A_j)}\right), \frac{\chi_B}{m(B)} \right\rangle. \end{aligned}$$

Let

$$J = \left\{ j\colon\ 1 \leqslant j \leqslant m, \operatorname{Re}\left[e^{i\theta}\left\langle \hat{h}_0\left(\frac{\chi_{A_j}}{\mu(A_j)}\right), \frac{\chi_B}{m(B)} \right\rangle \right] > 1-\epsilon^4 \right\}.$$

By Lemma 2.2 we have

$$\alpha_J = \sum_{j\in J} \alpha_j > 1 - \frac{\epsilon^8}{1-(1-\epsilon^4)} = 1-\epsilon^4.$$

We define

$$f_1 = \sum_{j\in J} \left(\frac{\alpha_j}{\alpha_J}\right) \frac{\chi_{A_j}}{\mu(A_j)}.$$

Then we can see $\|f_1\|_1 = 1$,

$$\begin{aligned} \|f_0 - f_1\|_1 &\leqslant \left\| \sum_{j\notin J} \alpha_j \frac{\chi_{A_j}}{\mu(A_j)} \right\|_1 + \left(\frac{1}{\alpha_J} - 1\right) \left\| \sum_{j\in J} \alpha_j \frac{\chi_{A_j}}{\mu(A_j)} \right\|_1 \\ &= \sum_{j\notin J} \alpha_j + (1-\alpha_J) = 2(1-\alpha_J) < 2\epsilon^4 \end{aligned}$$

and

$$\begin{aligned}\left|\left\langle \hat{h}_0(f_1), \frac{\chi_B}{m(B)}\right\rangle\right| &\geqslant \operatorname{Re}\left[e^{i\theta}\left\langle \hat{h}_0(f_1), \frac{\chi_B}{m(B)}\right\rangle\right] \\ &= \frac{1}{\alpha_J}\sum_{j\in J}\alpha_j \operatorname{Re}\left[e^{i\theta}\left\langle \hat{h}_0\left(\frac{\chi_{A_j}}{\mu(A_j)}\right), \frac{\chi_B}{m(B)}\right\rangle\right] \\ &> \frac{1}{\alpha_J}\sum_{j\in J}\alpha_j\left(1-\epsilon^4\right) = 1-\epsilon^4.\end{aligned}$$

On the other hand, for each $j \in J$

$$\begin{aligned}1-\epsilon^4 &< \operatorname{Re}\left[e^{i\theta}\left\langle \hat{h}_0\left(\frac{\chi_{A_j}}{\mu(A_j)}\right), \frac{\chi_B}{m(B)}\right\rangle\right] \\ &= \operatorname{Re}\left[e^{i\theta}\sum_{l=1}^{p} c_l \frac{(\mu\otimes m)(D_l\cap(A_j\times B))}{\mu(A_j)m(B)}\right] \\ &= \operatorname{Re}\left[e^{i\theta}\sum_{l=1}^{p} c_l\gamma_j\frac{\gamma_{j,l}}{\gamma_j}\right],\end{aligned}$$

where

$$\gamma_j = \sum_{l=1}^{p}\frac{(\mu\otimes m)(D_l\cap(A_j\times B))}{\mu(A_j)m(B)},$$

and

$$\gamma_{j,l} = \frac{(\mu\otimes m)(D_l\cap(A_j\times B))}{\mu(A_j)m(B)}.$$

We define

$$L = \left\{l\colon\ 1\leqslant l\leqslant p,\ \operatorname{Re}\left(e^{i\theta}c_l\right) > 1-\frac{\epsilon^2}{4}\right\},$$

and

$$L_j = \left\{l\colon\ 1\leqslant l\leqslant p,\ \operatorname{Re}\left(e^{i\theta}c_l\gamma_j\right) > 1-\frac{\epsilon^2}{4}\right\}.$$

For each $j \in J$ we can see $\gamma_j > 1-\epsilon^4$, and by Lemma 2.2 again

$$\sum_{l\in L_j}\frac{\gamma_{j,l}}{\gamma_j} > 1-\frac{\epsilon^4}{1-(1-\frac{\epsilon^2}{4})} = 1-4\epsilon^2.$$

Hence

$$\sum_{l\in L_j}\gamma_{j,l}>\bigl(1-4\epsilon^2\bigr)\bigl(1-\epsilon^4\bigr).$$

For every $j\in J$ we note that $L_j\subset L$ and

$$\sum_{l\in L}\frac{(\mu\otimes m)(D_l\cap(A_j\times B))}{\mu(A_j)m(B)}\geqslant\sum_{l\in L_j}\frac{(\mu\otimes m)(D_l\cap(A_j\times B))}{\mu(A_j)m(B)}$$
$$=\sum_{l\in L_j}\gamma_{j,l}>\bigl(1-4\epsilon^2\bigr)\bigl(1-\epsilon^4\bigr).$$

Set $D=\bigcup_{l\in L}D_l$.

Therefore

$$\left\langle\hat{\chi}_D(f_1),\frac{\chi_B}{m(B)}\right\rangle=\sum_{j\in J}\left(\frac{\alpha_j}{\alpha_J}\right)\cdot\sum_{l\in L}\frac{\mu\otimes m(D_l\cap(A_j\times B))}{\mu(A_j)m(B)}$$
$$\geqslant\sum_{j\in J}\left(\frac{\alpha_j}{\alpha_J}\right)\bigl(1-4\epsilon^2\bigr)\bigl(1-\epsilon^4\bigr)=\bigl(1-4\epsilon^2\bigr)\bigl(1-\epsilon^4\bigr)$$
$$>1-5\epsilon^2>1-\epsilon.$$

By Lemma 2.1 there is $g_0\in S_{L_1(\mu)}$ such that $\|(\hat{\chi}_D+\hat{\varphi})(g_0)\|_\infty=1$ and $\|f_1-g_0\|<\frac{4\epsilon}{1-\epsilon}$, where φ is any simple function in $L_\infty(\mu\otimes m)$ such that $\|\varphi\|_\infty\leqslant 1$ and φ vanishes on D. Therefore, we have

$$\|f_0-g_0\|_1\leqslant\|f_0-f_1\|_1+\|f_1-g_0\|_1\leqslant 2\epsilon^4+\frac{4\epsilon}{1-\epsilon}.$$

Define

$$h_1=e^{-i\theta}\chi_D+\sum_{l\notin L}c_l\,\chi_{D_l}\in L_\infty(\mu\otimes m).$$

Let S be the operator in $\mathcal{L}(L_1(\mu),L_\infty(m))$ corresponding to h_1. Then

$$\bigl\|S(g_0)\bigr\|_\infty=\bigl\|\hat{h}_1(g_0)\bigr\|_\infty=1$$

and

$$\|h_0-h_1\|_\infty=\max_{l\in L}\bigl|c_l-e^{-i\theta}\bigr|=\max_{l\in L}\bigl|e^{i\theta}c_l-1\bigr|.$$

However, $\operatorname{Re}(e^{i\theta}c_l)>1-\frac{\epsilon^2}{4}$ for every $l\in L$, hence

$$\bigl(\operatorname{Im}\bigl(e^{i\theta}c_l\bigr)\bigr)^2\leqslant 1-\bigl(\operatorname{Re}\bigl(e^{i\theta}c_l\bigr)\bigr)^2$$
$$<1-\left(1-\frac{\epsilon^2}{4}\right)^2=\frac{\epsilon^2}{2}-\frac{\epsilon^4}{16}.$$

Since

$$\left|e^{i\theta}c_l - 1\right| = \sqrt{\big(1 - \mathrm{Re}\big(e^{i\theta}c_l\big)\big)^2 + \big(\mathrm{Im}\big(e^{i\theta}c_l\big)\big)^2} < \sqrt{\epsilon^4/16 + \big(\epsilon^2/2 - \epsilon^4/16\big)} = \frac{\epsilon}{\sqrt{2}},$$

we conclude

$$\|h_0 - h_1\|_\infty < \frac{\epsilon}{\sqrt{2}},$$

hence

$$\|T - S\|_\infty \leqslant \|h - h_0\|_\infty + \|h_0 - h_1\|_\infty < \epsilon^8 + \frac{\epsilon}{\sqrt{2}} < \epsilon. \qquad \square$$

Let us observe that for the real Banach spaces $L_1(\mu)$ and $L_\infty(m)$ better estimates could be obtained by inspecting the above proof.

Theorem 2.4. *For the real Banach spaces $L_1(\mu)$ and $L_\infty(m)$, let T be a bounded operator from $L_1(\mu)$ into $L_\infty(m)$ such that $\|T\| = 1$. Given $0 < \epsilon < 1/5$ and $f_0 \in S_{L_1(\mu)}$ satisfying $\|T(f_0)\|_\infty > 1 - \epsilon^4$, there exist $S \in \mathcal{L}(L_1(\mu), L_\infty(m))$, $\|S\| = 1$ and $g_0 \in S_{L_1(\mu)}$ such that*

$$\big\|S(g_0)\big\|_\infty = 1, \qquad \|T - S\| < \epsilon \quad \textit{and} \quad \|f_0 - g_0\|_1 < 2\epsilon^2 + \frac{20\epsilon}{1 - 5\epsilon}.$$

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