AN INFINITE DIMENSIONAL VECTOR SPACE OF UNIVERSAL FUNCTIONS FOR H^∞ OF THE BALL

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Abstract. We show that there exists a closed infinite dimensional subspace of $H^\infty(B^n)$ such that every function of norm one is universal for some sequence of automorphisms of B^n.

1. Introduction

Consider the ball B^n in C^n, and a sequence (ϕ_k) of holomorphic automorphisms of B^n. A function $f \in H^\infty(B^n)$ of norm one is called universal for the ball of $H^\infty(B^n)$ relative to (ϕ_k) if the sequence $(f \circ \phi_k)$ is locally uniformly dense in ball$(H^\infty(B^n))$. Universality is often considered in Fréchet spaces of holomorphic functions, where the properties of norms of the functions is irrelevant. In this paper, however, bounding our functions is one of our primary concerns.

The study of universality actually began in 1914, and has been thoroughly documented in the work of K.-G. Grosse-Erdmann, [9]. We mention only a small part of the large body of literature on the subject here. In 1929, G. D. Birkhoff [2] proved that there exists an entire function f and a sequence (z_n) of points in C such that for each entire function g, there exists a subsequence (z_{n_j}) such that

$$f(z + z_{n_j}) \to g(z)$$

uniformly on compacta. In 1941, W. P. Seidel and J. L. Walsh [15] were able to prove a similar result on the unit disk, replacing the Euclidean translations by non-Euclidean ones. In 1955, M. Heins [10] established the existence of a Blaschke product B such that there exists a sequence (z_k) in $\mathbb{D}$ for which $|z_k| \to 1$ such that the functions

$$\{B(\frac{z + z_k}{1 + \overline{z_k} z})\}$$

are locally uniformly dense in the unit ball of $H^\infty(\mathbb{D})$. In fact, Mortini and the second author [7] show that for every sequence (z_k) tending to the boundary of the disk, there is a Blaschke product B such that $\{B((z+z_k)/(1+\overline{z_k}z))\}$ is dense in the unit ball of $H^\infty(\mathbb{D})$ (with the compact-open topology). Thus, given any sequence of points (z_k) in $\mathbb{D}$ that tends to the boundary, there exists a Blaschke product that is universal for the unit ball of $H^\infty(\mathbb{D})$ relative to the automorphisms defined by

$$L_{z_k}(z) = (z + z_k)/(1 + \overline{z_k} z).$$

For the ball B^n in C^n, Chee [3] has shown the following. If, for each positive integer k, we choose a pair of positive real numbers x_k and y_k satisfying

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$$x_k^2 + y_k^2 = 1, \tag{1.1}$$

then there exists a universal function in the unit ball of $H^\infty(B^n)$ relative to the automorphisms

$$L_k(w) = (\phi_{k1}(w), \dots, \phi_{kn}(w)), \text{ for } w \in B^n, \tag{1.2}$$

where

$$\phi_{k1}(w) = \frac{w_1 + x_k}{1 + x_k w_1}, \tag{1.3}$$

$$\phi_{kj}(w) = \frac{y_k w_j}{1 + x_k w_1}, \text{ for } 2 \le j \le n. \tag{1.4}$$

Gauthier and Xiao [4] showed that the universal function can be taken to be an inner function. We will choose our $x_k \to 1$, so that $z_k := L_k(0, \dots, 0) = (x_k, 0, ..., 0) \to (1, 0, \dots, 0)$, which is a peak point for the algebra of functions continuous on the closure of the ball and holomorphic inside the ball, and henceforth we will write L_{z_k} for the automorphism in (1.2) corresponding to the point z_k. Along these lines, A. Montes-Rodríguez has studied the question of which sequences of automorphisms give rise to universal functions for the space of holomorphic functions (see [13]).

Once we know that such universal functions exist, we can then ask for a measure of how many such functions there are. We might ask whether or not the set of universal functions is dense in the space, for example. This question has been considered, and results can be found in the aforementioned work of Grosse-Erdmann, [9]. In another measure of how many universal functions a space can have, Bernal-González and Montes-Rodríguez (see [5] and [12]) studied universal functions for the space of analytic functions on a complex region Ω (different from the punctured plane) and showed that for certain sequences of automorphisms $\{\phi_k\}$ there exists an infinite dimensional closed vector subspace of $H(\Omega)$ such that for all nonzero f in the subspace, the set $\{f \circ \phi_k\}$ is dense in $H(\Omega)$. It is this work that led us to the question of whether or not there is a closed infinite dimensional vector subspace of $H^\infty(B^n)$ such that every function in the space of norm one is universal for the ball of $H^\infty(B^n)$. We now turn to showing that the answer to this question is affirmative.

The following lemma can be found in [1] and is essential to what follows.

Lemma 1. *Let f_j and g_j satisfy $|f_j| + |g_j| \le 1$ on B^n. Then*

$$\sum_{j=1}^{m} \|\lambda_j f_j g_1 g_2 \dots g_{j-1}\| \le \max_j |\lambda_j|,$$

where $g_0 \equiv 1$.

The following useful lemma, which is not new and can be found in other work on interpolation (for example, [6]), can be proved by induction.

Lemma 2. *For complex numbers $\lambda_1, \dots, \lambda_n$ with $|\lambda_j| \le 1$ for all j we have $|\prod_{j=1}^n \lambda_j - 1| \le \sum_{j=1}^n |\lambda_j - 1|$.*

In what follows, we will need to do interpolation on sequences of points in B^n while assuring that the norm of the function that does the interpolation can be chosen to be at most one. Such sequences were studied in [6], where the notion of asymptotic interpolating sequences of type one *(ais of type 1)* was introduced. In [6], it was shown that every sequence of points (z_k) tending to the boundary of B^n

has a subsequence (z_{m_k}) that is ais of type 1 ; that is, for any sequence (λ_k) of points in $\mathbb{D}$, there exists a function $h \in H^\infty(B^n)$ of norm one such that

$$|h(z_{m_k}) - \lambda_k| \to 0,$$

as $k \to \infty$. However, the proof there relies on a result presented in [11]. Rather than simply referencing those results here, we offer a short direct proof in the two lemmas below.

Lemma 3. *Let (z_m) be a sequence of points converging to a point $z_0 \in \partial B^n$. Let f denote the peaking function given by*

$$f(z) = \frac{1 + \langle z, z_0 \rangle}{2},$$

and let $T_k : \mathbb{D} \to \mathbb{D}$ be defined by

$$T_k(z) = \frac{1 - \overline{f(z_k)}}{1 - f(z_k)} \left(\frac{z - f(z_k)}{1 - z\overline{f(z_k)}} \right).$$

Then there exists a subsequence (z_{m_k}) of (z_m) and a corresponding sequence of functions $J_k \in H^\infty(B^n)$ such that $\|J_k\| = 1$, $J_k(z_{m_k}) = 0$ and

$$|J_k(z_{m_l}) - 1| < \frac{1}{2^{k+l}},$$

for $l > k$.

Proof. By defining $J_1 = T_1 \circ f$ we obtain a function of norm one with $J_1(z_1) = 0$ and $J_1(z_0) = 1$. So begin with J_1 and choose z_{m_2} so that $|J_1(z_{m_2}) - 1| < 1/2^3$. Now let $J_2 = T_{m_2} \circ f$. Then $J_2(z_{m_2}) = 0$ and $J_2(z_0) = 1$. So we may choose z_{m_3} so that

$$|J_k(z_{m_3}) - 1| < 1/2^{k+3}$$

for $k = 1, 2$. Proceeding in this way, we obtain the desired sequences. □

Lemma 4. *Let (z_k) be a sequence of points in B^n tending to the boundary. Then there exists a subsequence (z_{m_k}) that is* ais *of type 1.*

Proof. Let (z_k) be a sequence of points in B^n that tends to the boundary and let (λ_k) be a sequence of points in $\mathbb{D}$. Without loss of generality, we may pass to a subsequence converging to a point z_0 on the boundary. Since every boundary point is a peak point, we may construct functions J_k as in Lemma 3. Thus we obtain functions J_k of norm one such that $J_j(z_j) = 0$ and $|J_j(z_k) - 1| < 1/2^{j+k}$ for $k > j$.

By our assumption, we know that there exists a peak function $P_1 \in H^\infty(B^n)$, continuous on $\overline{B^n}$, such that $\|P_1\| = 1$, $P_1(z_0) = 1$ and $|P_1(z_m)| < 1$ for all m. Composing with an automorphism of $\mathbb{D}$ we may further assume that $P_1(z_1) \approx -1$ while $P_1(z_0) = 1$. Let $F_1 = [(P_1 - 1)/2]^2$ and $G_1 = [(P_1 + 1)/2]^2$. Then

$$F_1(z_0) = 0 \text{ and } F_1(z_1) \approx 1,$$

while

$$G_1(z_0) = 1 \text{ and } G_1(z_1) \approx 0,$$

and a quick computation shows that $|F_1| + |G_1| \leq 1$. In order to make our construction work, we will have to make sure that F_j tends to zero on future sequences and G_j tends to one. Since $F_1(z_0) = 0$ we may choose z_2 so that $|F_1(z_2)| < 1/2^{1+2}$ and since $G_1(z_0) = 1$ we may also assume that $|G_1(z_2) - 1| < 1/2^{1+2}$. Proceeding in this way we obtain sequences of functions F_j and G_j such that $F_j = [(P_j - 1)/2]^2$

and $G_j = [(P_j + 1)/2]^2$, where $\|P_j\| = 1$, $P_j(z_0) = 1$, $P_j(z_m) \to 1$ as $m \to \infty$, and $P_j(z_j) \to -1$ as $j \to \infty$.

Thus, we may assume that (z_j), F_j and G_j satisfy

1) $|F_j| + |G_j| \leq 1$;
2) $|F_j(z_j) - 1| < 1/2^j$ while $G_j(z_j) \to 0$ as $j \to \infty$;
3) For $k > j$ we have $|F_j(z_k)| < 1/2^{j+k}$ and $|G_j(z_k) - 1| < 1/2^{j+k}$.

For each $m \in \mathbb{N}$, let

$$h_m = \lambda_1 F_1 + \lambda_2 J_1 F_2 G_1 + \cdots + \lambda_m J_1 J_2 \cdots J_{m-1} F_m G_1 \cdots G_{m-1}.$$

By Lemma 1, we know that $\|h_m\| \leq 1$. Let $k \in \mathbb{N}$. Then we have $J_k(z_k) = 0$ and therefore , for $k \leq m$,

$$h_m(z_k) = (\lambda_1 F_1 + \lambda_2 J_1 F_2 G_1 + \cdots + \lambda_k J_1 J_2 \cdots J_{k-1} F_k G_1 \cdots G_{k-1})(z_k).$$

Now for $j = 1, \ldots, k-1$ we have $|F_j(z_k)| < 1/2^{j+k}$. Therefore, the sum of the first $k-1$ terms is smaller than $1/2^k$. Thus for $k \leq m$,

$$|h_m(z_k) - \lambda_k (J_1 J_2 \cdots J_{k-1} F_k G_1 \cdots G_{k-1})(z_k)| < 1/2^k.$$

So, using Lemma 2, the fact that $|J_j(z_k) - 1| < 1/2^{j+k}$ for $j < k$, and (2) and (3) above, we get

$$\begin{array}{rl} |\lambda_k (J_1 J_2 \cdots J_{k-1} F_k G_1 \cdots G_{k-1})(z_k) - \lambda_k| & \leq \\ \displaystyle\sum_{j=1}^{k-1} |J_j(z_k) - 1| + |F_k(z_k) - 1| + \sum_{j=1}^{k-1} |G_j(z_k) - 1| & \leq \\ 1/2^k + 1/2^k + 1/2^k. & \end{array}$$

So, $|h_m(z_k) - \lambda_k| < 1/2^{k-2}$ for $k \leq m$.

Now a normal families argument shows there is a function $h \in H^\infty(B^n)$ such that (a subsequence of) h_m tends to h uniformly on compacta; in particular, h has norm one. Now fix k and let $\epsilon > 0$. Then there exists $M > k$ such that $|h_m(z_k) - h(z_k)| < \epsilon$ for $m \geq M$. Therefore $|h(z_k) - \lambda_k| \leq |h(z_k) - h_m(z_k)| + |h_m(z_k) - \lambda_k| < \epsilon + 1/2^{k-2}$. Since ϵ is arbitrary, $|h(z_k) - \lambda_k| \to 0$ as $k \to \infty$.

□

Our work will be simplified by the following lemma, which we learned of from the paper by Gauthier and Xiao [4, Lemma 1].

Lemma 5. *If a sequence of holomorphic functions on a domain is bounded by* 1 *and converges at some point to a value of modulus* 1*, then the sequence converges to this value uniformly on compact subsets of the domain.*

Our main result is Theorem 7 in which we show that there is a closed, infinite dimensional vector space V of universal functions relative to a certain sequence of automorphisms. The space V will be constructed using the functions f_j and g_j that appear in Lemma 6, below.

Lemma 6. *For every sequence (z_k) in B^n tending to the boundary there exist a subsequence (z_m), automorphisms (T_m) of B^n and sequences (f_j) and (g_j) of functions in $H^\infty(B^n)$ with the following properties:*

1) (z_m) *can be written as a disjoint union of sequences; that is,* $(z_m) = \cup_k(z_{m,k})$,
2) $|f_j| + |g_j| \leq 1$, *and*
3) *For each* j *we have*

$$f_j \circ T_{z_{m,j}} \to 1 \text{ and } g_j \circ T_{z_{m,j}} \to 0 \text{ uniformly on compacta as } m \to \infty$$

and for $k \neq j$ *we have*

$$f_j \circ T_{z_{m,k}} \to 0 \text{ and } g_j \circ T_{z_{m,k}} \to 1 \text{ uniformly on compacta as } m \to \infty.$$

Proof. We first recall that B^n is transitive for every n (see, e.g., [14]), and so for every point $z_0 \in B^n$ there is an automorphism T_{z_0} taking 0 to z_0.

We apply Lemma 4 to obtain functions $h_k \in H^\infty(B^n)$ such that

$$h_j \circ T_{z_{m_j}}(0) \to 1$$

and, for $j \neq k$ we have

$$h_j \circ T_{z_{m_k}}(0) \to -1,$$

as $m \to \infty$.

By Lemma 5, we may conclude that

$$h_j \circ T_{z_{m,j}} \to 1 \text{ uniformly on compacta,}$$

while

$$h_j \circ T_{z_{m,k}} \to -1 \text{ uniformly on compacta,}$$

for $k \neq j$.

Let

$$f_j = \left(\frac{h_j + 1}{2}\right)^2 \text{ and } g_j = \left(\frac{h_j - 1}{2}\right)^2,$$

Then $|f_j| + |g_j| \leq 1$ and

(1) $f_j \circ T_{z_{m,j}}$ tends to 1 uniformly on compacta, while $g_j \circ T_{z_{m,j}}$ tends uniformly to 0 ;
(2) $f_j \circ T_{z_{m,k}}$ tends to 0 uniformly on compacta for $k \neq j$, while $g_j \circ T_{z_{m,k}}$ tends uniformly to 1.

□

We remind the reader that the automorphisms that we will be using in Theorem 7, below, are L_{z_k}, given by $L_{z_k}(z) = (z + z_k)/(1 + \overline{z_k} z)$ for $z \in \mathbb{D}$ and by $L_{z_k}(z) = (\phi_{k1}(z), ..., \phi_{kn}(z))$ for $z \in B^n$, where the ϕ_{kj} are chosen according to (1.3) and (1.4) and $x_k^2 + y_k^2 = 1$.

Theorem 7. *There exists a sequence of points* (z_m) *in* B^n *such that the space* $H^\infty(B^n)$ *contains a closed infinite dimensional vector space* V *with the property that every function of norm one in* V *is a universal function for the ball of* $H^\infty(B^n)$ *corresponding to* (L_{z_m}).

Proof. In the case of the unit disc, our sequence (z_m) can be chosen to be any ais of type 1. In the case of the ball, we choose any sequence (z_k) with $z_k = x_k + iy_k$, where x_k and y_k are positive and satisfy condition (1.1) with $x_k \to 1$, and use Lemma 4 to obtain a subsequence that is ais of type 1. We note that by Chee's result [3], there is a universal function associated with every sequence of this type, and therefore with every subsequence of our sequence.

Now write (z_m) as the disjoint union of countably many sequences; say, $(z_m) = \cup_k (z_{m,k} : m \in \mathbb{N})$. By our choice of the sequence, for each k there exists a universal function h_k of norm one (which can be taken to be inner) corresponding to $(L_{z_{m,k}})$. Let f_j and g_j be the functions given by Lemma 6, and consider the closed vector space given by

$$V = \overline{\text{Span}}\{h_1 f_1, h_2 f_2 g_1, h_3 f_3 g_1 g_2, \ldots\}.$$

We claim that V is infinite dimensional and every function in V of norm 1 is universal for the ball of $H^\infty(B^n)$.

So, let h be an element in V of norm one, let ϵ be chosen satisfying $0 < \epsilon < 1/2$, and choose $\lambda_1, \ldots, \lambda_r \in C$ so that

$$\|h - \sum_{k=1}^{r} \lambda_k h_k f_k g_1 \cdots g_{k-1}\| < \epsilon/4.$$

In order to show that h is universal, it will be necessary to estimate $\|\lambda_j\|_{\max}$.

We first claim that $\|\lambda_j\|_{\max} \leq 1 + 4\epsilon$. To see this, let j be an arbitrary integer between 1 and r, and let $\lambda = \max_{k=1}^{r} \{|\lambda_k|, 1\}$. Using the universality of h_j with respect to the sequence $(L_{z_{m,j}})$ and (3) in Lemma 6, for any fixed point $z \in D$ we can find m such that

$$|1 - h_j(L_{z_{m,j}}(z))| < \epsilon/(8\lambda), \ |1 - f_j(L_{z_{m,j}}(z))| < \epsilon/(8\lambda),$$
$$\text{and } |g_j(L_{z_{m,j}}(z))| < \epsilon/(4\lambda r),$$

while at the same time assuring that for $k = 1, \ldots, r$ with $k \neq j$ we also have

$$|f_j(L_{z_{m,k}}(z))| < \epsilon/(4\lambda r) \text{ and } |1 - g_j(L_{z_{m,k}}(z))| < \epsilon/(4\lambda r).$$

Now, the fact that $|f_k(z)| + |g_k(z)| \leq 1$ for all k and z implies that $\|f_k\| \leq 1$ and $\|g_k\| \leq 1$ for all k. Consequently,

$$|h(L_{z_{m,j}}(z)) - \sum_{k=1}^{r} \lambda_k h_k f_k g_1 \ldots g_{k-1}(L_{z_{m,j}}(z))| < \epsilon/4$$

implies that

$$|h(L_{z_{m,j}}(z)) - \lambda_j (h_j f_j g_1 \cdots g_{j-1})(L_{z_{m,j}}(z))| < \epsilon/2.$$

Since $\|h\| \leq 1$,

$$|\lambda_j (h_j f_j g_1 \cdots g_{j-1})(L_{z_{m,j}}(z))| < 1 + \epsilon/2. \quad (1.5)$$

Now, by Lemma 2,

$$|1 - h_j(L_{z_{m,j}}(z)) f_j(L_{z_{m,j}}(z)) g_1(L_{z_{m,j}}(z)) \cdots g_{j-1}(L_{z_{m,j}}(z))| \leq$$
$$|1 - h_j(L_{z_{m,j}}(z))| + |1 - f_j(L_{z_{m,j}}(z))| + \sum_{k=1}^{j-1} |g_k(L_{z_{m,j}}(z)) - 1| \leq \epsilon/(2\lambda), \quad (1.6).$$

Therefore returning to (1.5) we see that

$$|\lambda_j \Big(h_j(L_{z_{m,j}}(z)) f_j(L_{z_{m,j}}(z)) g_1(L_{z_{m,j}}(z)) \cdots g_{j-1}(L_{z_{m,j}}(z)) - 1 + 1 \Big)| \leq 1 + \epsilon/2,$$

which together with (1.6) implies that

$$|\lambda_j|(1 - \epsilon/2) \leq |\lambda_j|(1 - \epsilon/(2\lambda)) \leq 1 + \epsilon/2.$$

Thus $|\lambda_j| \leq 1 + 4\epsilon$ for $\epsilon < 1/2$, completing the proof of the claim.

In addition, by Lemma 1 we have $\|\lambda_j\|_{\max} > 1-\epsilon$. So we may choose j_0 such that $|\lambda_{j_0}| > 1-\epsilon$.

Next we show that h is universal. Let f be a function in the ball of $H^\infty(B^n)$ and let $K \subset B^n$ be a compact set. Write $\lambda_{j_0} = r_0 e^{i\theta_0}$. Since h_{j_0} is universal for the sequence of automorphisms corresponding to (z_{m,j_0}), there exists a subsequence (z_{m_k,j_0}) such that $\|h_{j_0} \circ L_{z_{m_k,j_0}} - e^{-i\theta_0} f\|_K \to 0$. Thus, recalling that $r_0 = |\lambda_{j_0}| > 1-\epsilon$ we get

$$\begin{aligned} \|\lambda_{j_0} h_{j_0} \circ L_{z_{m_k,j_0}} - f\|_K &\leq \\ \|h_{j_0} \circ L_{z_{m_k,j_0}} - e^{-i\theta_0} f\|_K + |1-r_0| &\leq \\ \|h_{j_0} \circ L_{z_{m_k,j_0}} - e^{-i\theta_0} f\|_K + \epsilon. \end{aligned}$$

Thus

$$\|\lambda_{j_0} h_{j_0} \circ L_{z_{m_k,j_0}} - f\|_K \leq \|h_{j_0} \circ L_{z_{m_k,j_0}} - e^{-i\theta_0} f\|_K + \epsilon. \tag{1.7}$$

Since K is a compact set, we may choose M so large that for $k \geq M$ three things happen:

1) By the preceding work, we may assume that $\|\lambda_{j_0} h_{j_0} \circ L_{z_{m_k,j_0}} - f\|_K < 2\epsilon$;
2) By Lemma 6 we may assume that for $j = 1, \ldots, j_0 - 1$ we have

$$\|f_j \circ L_{z_{m_k,j_0}}\|_K < \epsilon/2^j \text{ and } \|1 - g_j \circ L_{z_{m_k,j_0}}\|_K < \epsilon/2^j; \text{ and}$$

3) $\|1 - f_{j_0} \circ L_{z_{m_k,j_0}}\|_K \leq \epsilon$ and $\|g_{j_0} \circ L_{z_{m_k,j_0}}\|_K \leq \epsilon/r$.

By (2) above and our estimates on $|\lambda_j|$ we get

$$\|\sum_{j=1}^{j_0-1} \lambda_j h_j f_j g_1 \cdots g_{j-1} \circ L_{z_{m_k,j_0}}\|_K \leq$$

$$(1+4\epsilon) \sum_{j=1}^{j_0-1} \|f_j \circ L_{z_{m_k},j_0}\|_K \leq (1+4\epsilon)\epsilon < 3\epsilon. \tag{1.8}$$

By (3) above and our estimates on $|\lambda_j|$ we get

$$\|\sum_{j=j_0+1}^{r} \lambda_j h_j g_1 \cdots g_{j-1} \circ L_{z_{m_k,j_0}}\|_K \leq$$

$$(1+4\epsilon) r \|g_{j_0} \circ L_{z_{m_k,j_0}}\|_K \leq (1+4\epsilon)\epsilon < 3\epsilon. \tag{1.9}$$

So consider $h \circ L_{z_{m,j_0}}$ and recall that

$$\|h \circ L_{z_{m,j_0}} - \sum_{j=1}^{r} \lambda_j h_j f_j g_1 \cdots g_{j-1} \circ L_{z_{m,j_0}}\| < \epsilon/2.$$

Now, using our previous calculations we get

$$
\begin{aligned}
\|h\circ L_{z_{m_k},j_0} - f\|_K &\leq \\
\|\sum_{j=1}^{r} \lambda_j h_j f_j g_1 \ldots g_{j-1} \circ L_{z_{m_k},j_0} - f\|_K + \epsilon &\leq \\
\Big(\sum_{j\neq j_0} \|\lambda_j h_j f_j g_1 \cdots g_{j-1} \circ L_{z_{m_k},j_0}\|_K &+ \\
\|\lambda_{j_0}(h_{j_0} f_{j_0} g_1 \cdots g_{j_0-1}) \circ L_{z_{m_k},j_0} - f\|_K \Big) + \epsilon &= \\
\Big(\sum_{j<j_0} \|\lambda_j h_j f_j g_1 \cdots g_{j-1} \circ L_{z_{m_k},j_0}\|_K &+ \\
\|\lambda_{j_0}(h_{j_0} f_{j_0} g_1 \cdots g_{j_0-1}) \circ L_{z_{m_k},j_0} - f\|_K &+ \\
\sum_{j>j_0} \|\lambda_j (h_j f_j g_1 \ldots g_{j-1}) \circ L_{z_{m_k},j_0}\|_K \Big) + \epsilon &\leq \\
\|\lambda_{j_0}(h_{j_0} f_{j_0} g_1 \cdots g_{j_0-1}) \circ L_{z_{m_k},j_0} - f\|_K + 7\epsilon,
\end{aligned}
$$

where the penultimate inequality follows from (1.8) and (1.9).

Now, using Lemma 2, (1.7), as well as (1), and (2) and (3) listed above in this proof we obtain

$$
\begin{aligned}
\|\lambda_{j_0}(h_{j_0}\circ L_{z_{m_k},j_0})(f_{j_0} g_1 \cdots g_{j_0-1}) \circ L_{z_{m_k},j_0} - f\|_K \leq \\
\Big(\|\lambda_{j_0}(h_{j_0}\circ L_{z_{m_k},j_0}) - f\|_K + |\lambda_{j_0}| \|f_{j_0}\circ L_{z_{m_k},j_0} - 1\|_K + \\
|\lambda_{j_0}| \sum_{j=1}^{j_0-1} \|g_j \circ L_{z_{m_k},j_0} - 1\|_K \Big) \leq \\
\epsilon + (1+4\epsilon)\epsilon + (1+4\epsilon)\sum_{j=1}^{j_0} \epsilon/2^j \leq \\
8\epsilon.
\end{aligned}
$$

Therefore, we have

$$\|h\circ L_{z_{m_k},j_0} - f\|_K \leq 15\epsilon,$$

completing the proof that h is universal for the ball of $H^\infty(B^n)$.

It remains to show that V is infinite dimensional. So, suppose that we have $\lambda_1, \ldots, \lambda_p \in \mathbb{D}$ and

$$\sum_{k=1}^{p} \lambda_k h_k f_k g_1 \cdots g_{k-1} = 0.$$

We will assume that $\lambda_p \neq 0$ and obtain a contradiction. Passing to subsequences if necessary and using (2) and (3) above, we see that on compacta,

$$f_p \circ L_{z_{m,p}} \to 1,$$

and for $k \neq p$ we know

$$f_k \circ L_{z_{m,p}} \to 0$$

and

$$g_k \circ L_{z_{m,p}} \to 1.$$

As a consequence, we see that $\lambda_p h_p f_p g_1 \cdots g_{p-1} \circ L_{z_{m,p}} \to 0$ as $m \to \infty$ on compacta. Since h_p is universal, it must be that $\lambda_p = 0$, which concludes the proof. □

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