Lineability and spaceability of sets of functions on $\mathbb{R}$

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Abstract

We show that there is an infinite dimensional vector space of differentiable functions on $\mathbb{R}$, every non-zero element of which is nowhere monotone. We also show that there is a vector space of dimension 2^c of functions $\mathbb{R} \to \mathbb{R}$, every non-zero element of which is everywhere surjective.

1 Introduction

This article is a contribution to an ongoing search for what are often large vector spaces of functions on $[0, 1]$ or on $\mathbb{R}$ which have special properties. Given such a property, we say that the subset M of functions on $[0, 1]$ which satisfy it is *spaceable* if $M \cup \{0\}$ contains a *closed* infinite dimensional subspace. The set M will be called *lineable* if $M \cup \{0\}$ contains an infinite dimensional vector space. At times, we will be more specific, referring to the set M as $\mu-$lineable if it contains a vector space of dimension μ. Also, we let $\lambda(M)$ be the maximum cardinality (if it exists) of such a vector space.

One of the earliest results in this direction was proved by the second author [5], see also [7], who showed that the set of nowhere differentiable functions on $[0, 1]$ is lineable. Soon after, V. Fonf, V. Kadeč and the second author [3] showed that the set of *nowhere* differentiable functions on $[0, 1]$ is spaceable; that is, there is a closed, infinite dimensional subspace $X \subset \mathcal{C}[0, 1]$, the Banach space of continuous functions on $[0, 1]$, every non-zero element of which is nowhere differentiable on $[0, 1]$. In fact, much more is true. L. Rodríguez-Piazza showed that the X in [3] can be chosen to be isometrically isomorphic to any separable Banach space [12]. Several years ago, S. Hencl [9] showed that any separable Banach space is isometrically isomorphic to a subspace of $\mathcal{C}[0, 1]$ whose non-zero elements are nowhere approximately differentiable and nowhere Hölder. It is clear that the set of *everywhere* differentiable functions on $[0, 1]$ is linear and

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hence lineable. The second author showed in [5] that this cannot be improved: the set of everywhere differentiable functions on $[0, 1]$ is not spaceable. Recently, P. Enflo and the second author have shown [2] that for any infinite dimensional subspace $X \subset \mathcal{C}[0, 1]$, the set of functions in X having infinitely many zeros in $[0, 1]$ is spaceable in X.
Our work here continues this type of 'bad news-good news' program. Our main results are the following:
Theorem 3.4. The set $\mathcal{DNM}(\mathbb{R})$ of differentiable functions on $\mathbb{R}$ which are nowhere monotone is lineable in $\mathcal{C}(\mathbb{R})$.
In order to state our second main result, we need some notation: Let us agree to denote by $\mathcal{F}(\mathbb{R})$ the set of all functions $f : \mathbb{R} \to \mathbb{R}$ which are 'everywhere surjective;' that is, for any non-trivial interval $(a, b) \subset \mathbb{R}$, $f(a, b) = \mathbb{R}$. The existence of such functions f was noticed by H. Lebesgue in [11] (see also [4]). In fact, the point of the theorem below is that such functions are quite plentiful, in a very strong way.
Theorem 4.3. The set $\mathcal{F}(\mathbb{R})$ is 2^c-lineable.
Our argument for Theorem 4.3 will appeal to some set theoretic considerations. Also, the interested reader should refer to work of Katznelson and Stromberg [10] in connection with Theorem 3.4.

2 Fat functions and some problems of differentiability

The main tool in the proof of Theorem 3.4 will be pointwise analogues of classical results on differentiation of uniformly convergent series. We believe that these analogues may be of independent interest.

Definition 2.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is integrable on each finite subinterval. We say that f is $H-$**fat** $(0 < H < \infty)$ if for each $a < b$,*

$$\frac{1}{b-a} \cdot \left| \int_a^b f(t)dt \right| \leq H \cdot min\{|f(a)|, |f(b)|\} \tag{1}$$

*$H_f = inf(H)$ in (1) will be called the **fatness** of f. We say that f is **fat** if it is H-fat for some $H \in (0, \infty)$. A family $\mathcal{F}$ of such functions $\{f\}$ be called uniformly fat if $H_{\mathcal{F}} = sup_{f \in \mathcal{F}}(H_f) < \infty$.*

Roughly speaking, if a function is fat then its average value on any interval cannot be very large compared to its values at the endpoints of the interval.

Definition 2.2 *A positive continuous even function φ on $\mathbb{R}$ that is decreasiing on $\mathbb{R}_+$ is called a **scaling function**.*

The next result gives a useful sufficient condition for fatness.

Proposition 2.3 *Given a scaling function φ, if for each $b > 0$*

$$\frac{1}{b} \cdot \int_0^b \varphi(t)dt \leq K \cdot \varphi(b), \tag{2}$$

then φ is fat and $H_\varphi \leq 2K$. For $0 < a < b$ in (1) one can take $H = K$.

PROOF The basic idea of the proof utilizes the fact that the average value of a scaling function φ over an interval $[0, b]$ is at least as big as its average over $[a, b]$, where $0 < a < b$.
By symmetry, there is no loss if we suppose that $|a| < b$. There are two cases to consider:

1. $-b < a \leq 0$. Then
$$\frac{1}{b-a} \cdot \int_a^b \varphi(t)dt \leq \frac{2}{b} \cdot \int_0^b \varphi(t)dt \leq 2K \cdot \varphi(b)$$

2. $0 < a < b$. Making the linear substitution $t = t(x) = a + \frac{b-a}{b}x$ we have $t(0) = a$, $t(b) = b$, so that we have $t(x) \geq x$ on $[0, b]$. Since φ is decreasing, $\varphi(t(x)) \leq \varphi(x)$. Therefore
$$\frac{1}{b-a} \cdot \int_a^b \varphi(t)dt = \frac{1}{b} \cdot \int_0^b \varphi[t(x)]dx \leq \frac{1}{b} \cdot \int_0^b \varphi(x)dx \leq K \cdot \varphi(b).$$

■

Example 2.4 *Each scaling function* $(1+|t|)^{-\alpha}$, $0 < \alpha < 1$ *is fat. In particular, for the function*
$$\varphi(t) = \frac{1}{\sqrt{1+|t|}} \tag{3}$$
we have that $H_\varphi \leq 4$.

Indeed
$$\frac{1}{b \cdot \varphi(b)} \cdot \int_0^b \frac{dx}{\sqrt{1+x}} = \frac{2(b+1-\sqrt{b+1})}{b} = 2 - \frac{\sqrt{b+1}-1}{b} < 2$$
So, $H_\varphi \leq 4$.

Definition 2.5 *For a scaling function* φ, *let* $L(\varphi)$ *denote the set of functions of the form*
$$\Psi(x) = \sum_{j=1}^{n} c_j \cdot \varphi(\lambda_j(x-\alpha_j)) \qquad \text{where} \qquad c_j, \lambda_j > 0, \quad \text{and} \quad \alpha_j \in \mathbb{R}. \tag{4}$$
For obvious reasons, the functions in $L(\varphi)$ *will be called of '*φ*-wavelet' type, or merely* φ*-functions.*

It is easy to see that the following is true:

Proposition 2.6 *If a scaling function* φ *is fat, then* $L(\varphi)$ *is uniformly fat. Moreover,* $H_{L(\varphi)} = H_\varphi$.

PROOF The proof follows by verifying that fatness of positive functions is preserved under translation, dilations, and positive linear combinations, and then using the description of the elements of $L(\varphi)$ given in (4).

■

Proposition 2.7 *[flexibility of $L(\varphi)$]* *Choose an arbitrary scaling function φ, $n \in \mathbb{N}$, n distinct real numbers $\{\alpha_j\}_{j=1}^n$, and intervals $\{I_j = (y_j, \tilde{y_j})\}_{j=1}^n$, where $0 < y_j < \tilde{y_j}$ for each $j = 1, 2, \ldots, n$. Then there exists $\psi \in L(\varphi)$ such that the following two conditions are satisfied:*

1. *$\psi(\alpha_j) \in I_j$ for $j = 1, 2, \ldots, n$.*
2. *$\psi(x) < max_{1 \leq j \leq n}\, \tilde{y_j}$ for all $x \in \mathbb{R}$.*

Proof For a given $\varepsilon > 0$, it is not difficult to construct an "ε-analogue" of the δ-function in $L(\varphi)$, namely a function $\varphi_\varepsilon(x) \in L(\varphi)$, such that $\varphi_\varepsilon(0) = 1$, $\varphi_\varepsilon(x) \leq 1$ for all $x \in \mathbb{R}$, and such that $\varphi_\varepsilon(x) < \varepsilon$ for $x \notin [-\varepsilon, \varepsilon]$. The result follows by judicious use of the family (4).

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The main tool here is based on the following "pointwise" analogue of classical "uniform" theorems on differentiation of series.

Theorem 2.8 *Let $\sum_{n=1}^\infty \Psi_n(x)$ be a formal series of continuously differentiable functions on $\mathbb{R}$, such that for some $x_0 \in \mathbb{R}$, $\sum_{n=1}^\infty \Psi_n(x_0)$ converges. For each n, let $\Psi'_n = \psi_n$ and suppose that $\{\psi_n \ : \ n \in \mathbb{N}\}$ is a uniformly fat sequence of positive functions, with $\sum_{n=1}^\infty \psi_n(a)$ converging to s, say, for some a. Then*

1. *$F(x) \equiv \sum_{n=1}^\infty \Psi_n(x)$ is uniformly convergent on each bounded subset of $\mathbb{R}$.*
2. *$F'(a)$ exists and $F'(a) = s$.*

In particular, if $\sum_{n=1}^\infty \psi_n(x) = s(x) < \infty$ for each $x \in \mathbb{R}^n$, then $F'(x) = s(x)$ for all $x \in \mathbb{R}$. (Of course $s(x)$ need not be continuous if the convergence is not uniform).

Proof Take $b > \max\{|a|, |x_0|\}$ and let

$$\tilde{\Psi}_n(x) = \int_0^x \psi_n(t)dt,\ n = 1, 2, \ldots$$

Since ψ_n is H-fat for some $H < \infty$, $n = 1, 2, \ldots$, we have for $|x| \leq b$

$$\left|\tilde{\Psi}_n(x)\right| \leq \left|\int_0^a \psi_n(t)dt\right| + \left|\int_a^x \psi_n(t)dt\right|$$

$$\leq H \cdot |a| \cdot \psi_n(a) + H \cdot |x - a| \cdot \psi_n(a) \leq 3Hb\psi_n(a)$$

So, by the Weierstrass M-test, $\sum_{n=1}^\infty \tilde{\Psi}_n(x)$ converges uniformly on $[-b, b]$ to some function $\tilde{F}$.
Next, since $\Psi_n(x) = \tilde{\Psi}_n(x) + C_n$ for each n and since $\sum_n \Psi_n(x_0)$ converges, it follows that $\sum_n C_n = C$ converges. Consequently, $\sum_{n=1}^\infty \Psi_n(x)$ is uniformly convergent on $[-b, b]$ to some function $F(x) = \tilde{F}(x) + C$, which proves 1.
Let $\varepsilon > 0$ be arbitrary, and choose N such that $\sum_{n=N+1}^\infty \psi_n(a) < \frac{\varepsilon}{2(H+1)}$. By the continuity of each ψ_n, there is some $\delta > 0$ such that for all $|h| < \delta$ and $n < N$,

$$\left|\frac{1}{h} \cdot \int_a^{a+h} \psi_n(t)dt - \psi_n(a)\right| < \frac{\varepsilon}{2N}$$

The uniform fatness of $\{\psi_n \ : \ n \in \mathbb{N}\}$ implies that

$$\left|\frac{F(a+h)-F(a)}{h}-s\right| = \left|\frac{\tilde{F}(a+h)-\tilde{F}(a)}{h}-s\right| = \left|\sum_{n=1}^{\infty}\left(\frac{1}{h}\cdot\int_a^{a+h}\psi_n(t)dt-\psi_n(a)\right)\right| \leq$$

$$\left|\sum_{n=1}^{N}\left(\frac{1}{h}\cdot\int_a^{a+h}\psi_n(t)dt-\psi_n(a)\right)\right| + \left|\sum_{n=N+1}^{\infty}\left(\frac{1}{h}\cdot\int_a^{a+h}\psi_n(t)dt-\psi_n(a)\right)\right| \leq$$

$$\frac{\varepsilon N}{2N} + \sum_{n=N+1}^{\infty}(H\psi_n(a)+\psi_n(a)) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

■

Remark: The assumption that each of the ψ_n's be non-negative can be avoided by instead requiring that $\sum_{n=1}^{\infty}|\psi_n(a)| < \infty$.

3 Existence of differentiable functions with some given properties of their derivatives

Our main results in this section are Theorems 3.4 and 3.5, which are consequences of the following two results.

Theorem 3.1 *Let $0 = y_0 < y_1 < y_2 < \ldots < y_n < \ldots \to 1$. Let $S_0 = \{\alpha_j^0\}_{j=1}^{\infty}$ be a countable set of distinct real numbers and, for each $i \in \mathbb{N}$, let $S_i = \{\alpha_j^{(i)}\}_{j=1}^{m_i}$ be a finite set of distinct real numbers. Suppose further that the sets $\{S_i\}_{i=0}^{\infty}$ are pairwise disjoint. Then, there exists a differentiable function F on $\mathbb{R}$ such that*

1. *$F'(\alpha_j^{(i)}) = y_j$ for all $j = 1, 2, ..., m_i$ and $i = 1, 2, \ldots$*
2. *$F'(\alpha_j^{(0)}) = 1$ for all $j \in \mathbb{N}$.*
3. *$0 < F'(x) \leq 1$, for all $x \in \mathbb{R}$.*

Proof For each i and each interval $I_i = (y_{i-1}, y_i)$, consider a strictly increasing sequence $(y_{i,j})$ such that $\{y_{ij}\} \in I_i$ and $\lim_{j\to\infty} y_{ij} = y_i$. Let φ be a fat scaling function on $\mathbb{R}$. By Proposition 2.7 there exists $f_1 = \psi_1 \in L(\varphi)$ such that:

I1. $\psi_1(\alpha_j^{(1)}) \in (y_{1,0}, y_{1,1})$ for $j = 1, 2, \ldots, m_1$.

I2. $\psi_1(\alpha_1^0) \in (y_{1,0}, y_{1,1})$, and

I3. $\psi_1(x) < y_{1,1}$ for all $x \in \mathbb{R}$.

By Proposition 2.7, we can choose $\psi_2 \in L(\varphi)$ such that if $f_2 = \psi_1 + \psi_2$, then the following hold:

II1. $f_2(\alpha_j^{(1)}) \in (y_{1,1}, y_{1,2})$, for $j = 1, 2, \ldots, m_1$.
$f_2(\alpha_j^{(2)}) \in (y_{2,1}, y_{2,2})$, for $j = 1, 2, \ldots, m_2$.

II2. $f_2(\alpha_j^0) \in (y_{2,1}, y_{2,2})$ for $j = 1, 2$, and

II3. $f_2(x) < y_{2,2}$, for all $x \in \mathbb{R}$.

Continuing we get a sequence (f_n), where $f_n = \sum_{i=1}^{n} \psi_i, n = 1, 2, \ldots,$ and where each $\psi_i \in L(\varphi)$ is such that the following conditions hold:

N1. $f_n(\alpha_j^{(1)}) \in (y_{1,n-1}, y_{1,n})$, for $j = 1, 2, \ldots, m_1$.
$f_n(\alpha_j^{(2)}) \in (y_{2,n-1}, y_{2,n})$, for $j = 1, 2, \ldots, m_2$.
$\vdots$
$f_n(\alpha_j^{(n)}) \in (y_{n,n-1}, y_{n,n})$, for $j = 1, 2, \ldots, m_n$.

N2. $f_n(\alpha_j^0) \in (y_{n,n-1}, y_{n,n})$ for $j = 1, 2, \ldots, n$.

N3. $f_n(x) < y_{n,n}$, for all $x \in \mathbb{R}$.

Since $f_n(x) \leq 1$ and $\psi_n(x) > 0$ for all x, the series $\psi(x) = \sum_{n=1}^{\infty} \psi_n(x)$ converges for all $x \in \mathbb{R}$.
It follows from Theorem 2.8 that the function $F(x) = \int_0^{\infty} \psi(x)dx$ satisfies all the assertions in the statement of the theorem.

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Remark: It would be interesting to determine whether a weaker condition than fatness suffices to yield the conclusions of Theorems 2.8 and 3.1.

The following is a simple consequence of Theorem 2.8.

Theorem 3.2 *Let A^+, A^-, A^0 be pairwise disjoint countable sets in $\mathbb{R}$. There exists a differentiable function F on $\mathbb{R}$ such that $F'(x) \leq 1$ for all $x \in \mathbb{R}$ and such that:*

1. $F'(x) > 0$, $x \in A^+$.
2. $F'(x) < 0$, $x \in A^-$.
3. $F'(x) = 0$, $x \in A^0$.

Proof By theorem 2.8 there exist two everywhere differentiable functions $F(x), G(x)$ such that

1. $F'(x) = 1$ for $x \in A^+ \cup A^0$, $F'(x) < 1$ for $x \in A^-$, and $0 < F'(x) \leq 1$ for $x \in \mathbb{R}$.
2. $G'(x) = 1$ for $x \in A^- \cup A^0$, $G'(x) < 1$ for $x \in A^+$, and $0 < G'(x) \leq 1$ for $x \in \mathbb{R}$.

Obviously, the function $H(x) = F(x) - G(x)$ satisfies the conditions of theorem 3.1.

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Theorem 3.2 easily implies the following classical result (see, e.g., [4]).

Corollary 3.3 *There exist everywhere differentiable nowhere monotone functions on $\mathbb{R}$.*

Proof Apply theorem 3.2 for A^+, A^- and A^0 dense in $\mathbb{R}$.

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Moreover, there are "linearly many" such functions:

Theorem 3.4 *The set $\mathcal{DNM}(\mathbb{R})$ is lineable in $C(\mathbb{R})$.*

PROOF Let's consider the sequence on triples of pairwise disjoint sets $\{A_k^+, A_k^-, A_k^0\}$ with the following properties:

1. Each of the three sets in each triple is dense in $\mathbb{R}$.
2. Each of the three sets in the triple $\{A_k^+, A_k^-, A_k^0\}$ is a subset of A_{k-1}^0.

By Theorem 3.1, for each k there exists an everywhere differentiable function $f_k(x)$ on $\mathbb{R}$ such that

1. $f_k(x) > 0,\ x \in A_k^+$.
2. $f_k(x) < 0,\ x \in A_k^-$.
3. $f_k(x) = 0,\ x \in A_k^0$.

Obviously each f_k is nowhere monotone and the sequence $\{f_k\}_1^\infty$ is linearly independent. To complete the proof, let us show that if $f = \sum_{k=1}^n \alpha_k f_k$, $\{\alpha_k\}_1^n$ not all zero, then f is nowhere monotone. Without loss, we may suppose that $\alpha_n \neq 0$. On A_n^+ all f_k vanish for $k < n$, and so $f = \alpha_n f_n$, which implies that f is nowhere monotone. This proves the lineability of $\mathcal{DNM}(\mathbb{R})$.

■

We conclude this section with a corollary, which follows directly from the previous results and [5], and some remarks.

Theorem 3.5 *For finite a, b the set $\mathcal{DNM}[a,b]$ is lineable and not spaceable in $C[a,b]$.*

This is an immediate consequence of Theorem 3.4 and [6].

Final Remarks: 1. An obvious question related to Theorem 3.5 is whether the set $\mathcal{DNM}(\mathbb{R})$ is spaceable in the Fréchet space $C(\mathbb{R})$, endowed with the topology of uniform convergence on closed intervals in $\mathbb{R}$. We conjecture that this is indeed the case.

2. There seem to be a number of interesting questions concerning the relation between *finite lineability* and *countably infinite lineability*. For instance, in [1], the authors show that for any $3-$homogeneous polynomial $P : X \to \mathbb{R}$ defined on a real Banach space X, $P^{-1}(0)$ is $n-$lineable for any n. However, it is unknown if $P^{-1}(0)$ contains an infinite dimensional subspace.
It is not difficult to provide natural examples of sets which are $n-$lineable for every n but which are not infinitely lineable. For instance, let $j_1 \leq k_1 < j_2 \leq \ldots \leq k_m < j_{m+1} \leq \ldots$ be integers and let $M = \cup_m \{\sum_{i=j_m}^{k_m} a_i x^i \ :\ a_i \in \mathbb{R}\}$. Since the sets $\{\{\sum_{i=j_m}^{k_m} a_i x^i \ :\ a_i \in \mathbb{R}\} \ :\ n \in \mathbb{N}\}$ are pairwise disjoint, M is finitely, but not infinitely, lineable in $\mathcal{C}[0,1]$. Depending on the choice of the sequence (j_n), M may even be closed in $\mathcal{C}[0,1]$. For instance, it is shown in [8] that if (j_n) is a lacunary sequence, then $\{(x_n^{j_n})\}$ is a basic sequence in $\mathcal{C}[0,1]$. On the other hand, no matter what sequence $j_1 \leq k_1 < j_2 < \ldots$ we take, the corresponding set of complex polynomials $M = \cup_n \{\sum_{\ell=j_n}^{k_n} a_\ell z^\ell \ :\ a_\ell \in \mathbb{C}\}$ is always closed in $\mathcal{H}^\infty$.

4 Everywhere surjective functions

H. Lebesgue [11] was perhaps the first to show a somewhat surprising example of a function $f : \mathbb{R} \to \mathbb{R}$ with the property that on every non-trivial interval (a, b), $f(a, b) = \mathbb{R}$. (A more modern reference is [4], where examples of other functions are indicated.) Our main goal in this section is to prove that the set of such everywhere surjective functions is lineable. In fact, the set of such functions contains a vector subspace of the largest possible dimension, 2^c.
In order to do this, we first show that the set of surjective functions $f : \mathbb{R} \to \mathbb{R}$ is 2^c-lineable. The proof makes use of the following simple observation:

Lemma 4.1 *Let $C_1, C_2, \ldots, C_m$ be distinct non-empty sets. Then there exists a $k \in \{1, 2, \ldots, m\}$ such that $C_k \setminus C_i \neq \emptyset$ for all $i \neq k$.*

PROOF Suppose that for every $k \in \{1, 2, \ldots, m\}$ there exists $i \neq k$ such that $C_k \setminus C_i = \emptyset$. By relabelling the sets, we would then have $C_1 \subset C_2 \subset C_3 \subset \cdots \subset C_{m-1} \subset C_m$, which is a contradiction.

■

Proposition 4.2 *There exists a vector space Λ of functions $\mathbb{R} \to \mathbb{R}$ with the following two properties:*
(i). Every non-zero element of Λ is an onto function, and
(ii). dim(Λ)= 2^c.

PROOF Let $r \in \mathbb{R}, r \neq 0$, be fixed. For a non-empty subset $C \subset \mathbb{R}$, define $H_C : \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}$ by

$$H_C(y, x_1, x_2, x_3, \ldots) = \varphi_r(y) \cdot \prod_{i=1}^{\infty} I_C(x_i).$$

Here, I_C is the indicator function and $\varphi_r(y) = e^{ry} - e^{-ry}$, the most important property of this function being the fact that for any $r \neq 0$, $\varphi_r : \mathbb{R} \to \mathbb{R}$ is onto.
First of all, we show that the family $\{H_C : \emptyset \neq C \subset \mathbb{R}\}$ is linearly independent. Indeed, let's take m distinct subsets $C_1, C_2, \cdots, C_m$ of $\mathbb{R}$. Suppose that for some choice of scalars, the function

$$\sum_{j=1}^{m} \lambda_j H_{C_j}$$

is identically 0. Clearly, we may assume that no λ_j is 0, and since all the C_j's are different we may further assume that for every $j < m$, there exists a point $x_j \in C_m \setminus C_j$. Evaluating at the point

$$\overline{x} = (1, x_1, x_2, \ldots, x_{m-2}, x_{m-1}, x_{m-1}, x_{m-1}, \ldots),$$

we obtain

$$0 = \sum_{j=1}^{m} \lambda_j H_{C_j}(\overline{x}) = \varphi_r(1) \cdot \sum_{j=1}^{m} \left[\lambda_j \cdot \prod_{i=1}^{\infty} H_{C_j}(x_i) \right] = \varphi_r(1) \cdot \lambda_m.$$

So, we get that $\lambda_m = 0$, which is a contradiction. Hence the family $\{H_A : A \subset \mathbb{R}, A \neq \emptyset\}$ is linearly independent.

We now show that every $h \in \Gamma = \text{span}\{H_A : A \subset \mathbb{R}, A \neq \emptyset\}$, $h \neq 0$, is onto. First, given $s \in \mathbb{R}$ choose $b \in \mathbb{R}$ such that $\varphi_r(b) = s$. Then $H_A(b, a, a, a, \ldots) = s$, where $a \in A$ is arbitrary, which shows that each H_A is onto. If $h \in \Gamma \setminus \{0\}$ then for some non-zero real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ and some distinct non-empty subsets of $\mathbb{R}$, $C_1, C_2, \ldots, C_m$, we have

$$h = \sum_{j=1}^{m} \lambda_j H_{C_j}.$$

Arguing as before, we first find some C_j (which, without loss, is C_m) such that for each $j = 1, ..., m-1$, there is a point $x_j \in C_m \backslash C_j$. Given $s \in \mathbb{R}$, let $\overline{x} = (a, x_1, x_2, \ldots, x_{m-2}, x_{m-1}, x_{m-1}, \ldots)$, where $\varphi_r(a) = \frac{s}{\lambda_m}$. It is straightforward to verify that $h(\overline{x}) = s$.
So, every $h \in \Gamma \setminus \{0\}$ is onto. It is clear that $\dim(\Gamma) = 2^c$ since $\text{Card}(\{A : A \subset \mathbb{R}, A \neq \emptyset\}) = 2^c$. Since there is a bijection between $\mathbb{R}$ and $\mathbb{R}^{\mathbb{N}}$, we can also construct a vector space Λ of onto functions $f : \mathbb{R} \to \mathbb{R}$ having dimension 2^c.

■

We now come to the main result of this section, that the set of so-called Lebesgue everywhere surjective functions is 2^c-lineable. Note that 2^c is the cardinality of all functions $\mathbb{R} \to \mathbb{R}$, so obviously this result is best possible.

Theorem 4.3 *The set* $\mathcal{F}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \ : \ \text{for every } (a,b) \subset \mathbb{R}, \ \ f(a,b) = \mathbb{R}\}$ *is* 2^c*-lineable.*

Proof Choose Λ as in the preceding proposition, and fix any everywhere surjective function f. We claim that the vector space $\Delta = \{H \circ f : H \in \Lambda\}$ satisfies the required conditions. First, $\dim(\Delta)= 2^c$ since $\dim(\Lambda)= 2^c$. To see this it is enough to show that given m linear independent functions $H_j \in \Lambda \setminus \{0\}$, $j = 1, \ldots, m$, then the family $\{H_j \circ f\}_{j=1}^m$ is also linear independent. Take m nonzero real numbers $\{\lambda_j\}_{j=1}^m$, and suppose that the function $h = \sum_{j=1}^m \lambda_j \cdot (H_j \circ f)$ is identically zero. By construction h can be written as $h = G \circ f$, where G is onto. Take any $0 \neq s \in \mathbb{R}$. There exists $d \in \mathbb{R}$ with $G(d) = s$, and there also exists $a \in \mathbb{R}$ so that $f(a) = d$. Thus $h(a) = s \neq 0$, which is a contradiction.

Next let's take $g \in \Delta \setminus \{0\}$, $s \in \mathbb{R}$, and any interval $(a,b) \subset \mathbb{R}$. We need to find $\ell \in (a,b)$ such that $g(\ell) = s$. We can express g as $g = G \circ f$, with G being an onto function. So, for $s \in \mathbb{R}$ there exists $d \in \mathbb{R}$ such that $G(d) = s$. Now, we can find $\ell \in (a,b)$ with $f(\ell) = d$ (since f is our everywhere surjective function). So we have

$$g(\ell) = (G \circ f)(\ell) = G(f(\ell)) = G(d) = s, \text{ and}$$

we are done.

■

In future work, the authors hope to investigate 'algebrability' of sets M of functions, that is, when does a set M of functions contain a large algebra? In particular, the question of whether there is a Banach algebra of nowhere differentiable functions on $[0,1]$ seems interesting.

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