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p-Compact homogeneous polynomials from an ideal point of view

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Abstract. We prove that the space of p-compact n-homogeneous polynomials is a composition ideal of polynomials and prove an even stronger ideal condition. An application to the study of the stability of $p-$compactness under the formation of projective symmetric tensor products is provided. We also show that an $n-$homogeneous polynomial is $p-$compact if and only if its transpose is quasi $p-$nuclear. This solves a problem posed in [**2**].

1. Introduction

The theory of p-compact operators was initiated by Sinha and Karn and interest in this area has grown in the last few years. This is due in part to the fact that p-compact operators turn out to be a Banach ideal of operators, denoted $[\mathcal{K}_p, k_p]$, whose norm k_p was introduced by Sinha and Karn [**17**] and characterized by Delgado, Piñeiro and Serrano [**6**, Proposition 3.15]. The position of $[\mathcal{K}_p, k_p]$ among classical Banach ideals was first studied in [**17**], where it was proved that any operator whose adjoint is p-compact is p-summing, $\mathcal{K}_p$ is contained in the ideal Π_p^d of operators with p-summing adjoint, and p-nuclear operators have adjoints that are p-compact. This study was deepened in [**6**], where it was shown that an operator T is quasi p-nuclear if and only if its adjoint T^* is p-compact. The authors of [**6**] also proved the related dual result: an operator is p-compact if and only if its adjoint is quasi p-nuclear. The ideal $[\mathcal{QN}_p, \nu_p^Q]$ of quasi p-nuclear operators, introduced by Persson and Pietsch [**13**], was shown to be an important tool in the study of p-nuclear operators and the approximation properties of order p [**15**]. The above characterizations show the strong relationship between p-compact operators and quasi p-nuclear operators. Moreover, the norms involved also display good behavior: $\nu_p^Q(T) = k_p(T^*)$, whereas $\nu_p^Q(T^*) \le k_p(T)$ for the related dual result (see [**6**, Propositions 3.1 and 3.2]). These results show that $\mathcal{K}_p$ has a natural place in operator ideal theory.

The concept of p-compact holomorphic mapping was introduced in [**2**] as a generalization of p-compact operators to the non-linear case. There, the relation

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between p-compact holomorphic mappings and their Taylor series expansions was discussed. Some topological aspects of these mappings were also analyzed.

A Banach operator ideal $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$ is *tensor stable* with respect to a tensor norm α if $T\otimes S$ belongs to $\mathcal{I}(E\hat{\otimes}_\alpha F; G\hat{\otimes}_\alpha H)$ whenever $T\in\mathcal{I}(E;G)$ and $S\in\mathcal{I}(F;H)$. As mentioned in [**5**], tensor stable ideals were first studied by Vala [**18**], who proved the ϵ-stability of compact operators. Holub [**10**] proved that absolutely $p-$summing operators are stable under the formation of injective tensor products and provided an example of absolutely summing maps whose projective tensor product is not $p-$absolutely summing for any $1\le p<\infty$. He also proved the α-stability of nuclear operators for any crossnorm $\epsilon\le\alpha\le\pi$. Stability of operators ideals has been also treated in [**5, 11, 14**].

In this paper, we look at p-compact m-homogeneous polynomials from an ideal point of view. We show that the space of p-compact n-homogeneous polynomials is a composition ideal of polynomials and prove an even stronger ideal condition (see Theorem 3.2). As an application, we study the stability of $p-$compact operators and $p-$compact polynomials under the formation of symmetric tensor products. We prove that $\otimes_m T:\hat{\otimes}^{m,s}_{\pi_s}E\to\hat{\otimes}^{m,s}_{\pi}F$ is $p-$compact whenever $T:E\to F$ is a $p-$compact operator (see Section 2 for notation). The analogous result for homogeneous polynomials is also obtained: $\otimes_m P:\hat{\otimes}^{m,s}_{\pi_s}E\to\hat{\otimes}^{m,s}_{\pi}F$ is $p-$compact whenever $P:E\to F$ is a $p-$compact $m-$homogeneous polynomial. The notion of transpose of an operator was extended in [**3**, Proposition 3.2], to m-homogeneous polynomials. Among other things, it was shown that an m-homogeneous polynomial P is compact if and only if its "linear transpose" P^* is compact. Influenced by the linear case studied in [**6**], the relationship between a p-compact polynomial and its transpose is established. This solves a problem posed in [**2**].

2. Preliminaries and notation

In the sequel, E denotes a Banach space, B_E its closed unit ball and E^* its topological dual. If $x\in E$ and $\epsilon>0$ then $B_\epsilon(x)$ is the open ball of center x and radius ϵ. For $1\le p<\infty$, p' is given by $\frac{1}{p}+\frac{1}{p'}=1$. Let $\ell_p(E)$ denote the space of all sequences $(x_n)_n$ in E that are strongly p-summable; in other words, $\|(x_n)_n\|_p=(\sum_{n=1}^\infty||x_n||^p)^{1/p}<\infty$. In Section 3 we will need the related space $\ell_p^w(E)$ of weakly $p-$summable sequences in E. This space is formed by all sequences $(x_n)_n$ in E such that $(\varphi(x_n))_n\in\ell_p$ for every $\varphi\in E^*$. A set $K\subset E$ is said to be *relatively $p-$compact* if there exists a sequence $(x_n)_n$ in $\ell_p(E)$ such that

$$K\subset\{\sum_{n=1}^\infty a_nx_n\colon (a_n)_n\in B_{\ell_{p'}}\}.$$

We will use the abbreviated notation p-conv$\{(x_n)_n\}$ to denote the set

$$\{\sum_{n=1}^\infty a_nx_n\colon (a_n)_n\in B_{\ell_{p'}}\},$$

calling it the *p-convex hull* of $(x_n)_n$.

It is natural that for $p=\infty$, $\infty-$compact sets are just compact sets. In this case, $K\subset\{\sum_{n=1}^\infty a_nx_n\colon (a_n)_n\in B_{\ell_1}\}$, for some sequence $(x_n)_n$ in $c_0(E)$ (see e.g. [**7**, Lemma VIII.3.2]).

Given a subset $A\subset E$, the closed absolutely convex hull of A is denoted by $\overline{\Gamma}(A)$. It is well-known that p-conv$\{(x_n)_n\}$ is absolutely convex and for $1<p<\infty$

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it is closed. Therefore, for any $1 \leq p < \infty$, $\overline{\Gamma}(A)$ is $p-$compact whenever A is relatively $p-$compact.

Let E and F be Banach spaces. We denote by $\mathcal{L}(^m E; F)$ the space of all continuous $m-$linear mappings from $E \times \cdots \times E$ into F. Whenever $m = 1$, $\mathcal{L}(^1 E; F) = \mathcal{L}(E; F)$ coincides with the usual space of continuous linear operators, and for $m = 0$ we agree that $\mathcal{L}(^0 E; F)$ is the space of constant mappings and is identified with F. A mapping $P : E \longrightarrow F$ is a continuous *m-homogeneous polynomial* if there is $A \in \mathcal{L}(^m E; F)$ such that $P(x) = A(x, \ldots, x)$ for all $x \in E$. Let $\mathcal{P}(^m E; F)$ denote the space of all continuous m-homogeneous polynomials from E to F, endowed with the usual sup norm. In general, we will not explicitly say that a polynomial is continuous since all polynomials considered will be assumed to be continuous.

A mapping $f : E \longrightarrow F$ is *holomorphic* if, for each $a \in E$, there are $r > 0$ and a sequence $(\hat{d}^m f(a))_m$ of elements in $\mathcal{P}(^m E; F)$ such that $f(x) = \sum_{m=0}^{\infty} \frac{1}{m!} \hat{d}^m f(a)(x - a)$ uniformly for $x \in B_r(a)$. The space of all such mappings is denoted by $\mathcal{H}(E; F)$. We shall denote $P_m f := \frac{1}{m!} \hat{d}^m f(0)$. For the general theory of homogeneous polynomials and holomorphic functions we refer to [**8**] or [**12**].

A holomorphic mapping $f \in \mathcal{H}(E; F)$ is said to be *$p-$compact* if for each $x \in E$ there is $\epsilon > 0$ such that $f(B_\epsilon(x))$ is relatively $p-$compact in F. Let $\mathcal{H}_{\mathcal{K}_p}(E; F)$ denote the space of all $p-$compact holomorphic mappings from E to F, and let $\mathcal{P}_{\mathcal{K}_p}(^m E; F) := \mathcal{H}_{\mathcal{K}_p}(E; F) \cap \mathcal{P}(^m E; F)$.

By [**2**, Proposition 3.3], an $m-$homogeneous polynomial P is $p-$compact if and only if $P(B_E)$ is relatively $p-$compact in F. In particular, for $m = 1$, the space $\mathcal{P}_{\mathcal{K}_p}(^1 E; F)$ coincides with the space $\mathcal{K}_p(E; F)$ of all $p-$compact linear operators from E to F. The norm

$$k_p(T) = \inf\{\|(x_n)_n\|_p\}$$

makes $\mathcal{K}_p$ a Banach ideal (see [**6**]), where the infimum is taken over all sequences $(x_n)_n \in \ell_p(F)$ such that $T(B_E) \subset p-\text{conv}\{(x_n)_n\}$.

Given a Banach operator ideal $[\mathcal{I}, \|\cdot\|_{\mathcal{I}}]$, the *composition ideal* of polynomials $\mathcal{I} \circ \mathcal{P}$ consists of all homogeneous polynomials P between Banach spaces that can be factored as $P = u \circ Q$ where Q is a homogeneous polynomial and u is a linear operator belonging to $\mathcal{I}$. For $m \in \mathbb{N}$ and Banach spaces E and F, the usual composition norm $\|\cdot\|_{\mathcal{I} \circ \mathcal{P}}$ of an m-homogeneous polynomial $P \in \mathcal{I} \circ \mathcal{P}(^m E; F)$ is given by

$$\|P\|_{\mathcal{I} \circ \mathcal{P}} := \inf\{\|u\|_{\mathcal{I}} \|Q\| : P = u \circ Q, Q \in \mathcal{P}(^m E; G), u \in \mathcal{I}(G; F)\}. \tag{2.1}$$

With this norm $\mathcal{I} \circ \mathcal{P}$ becomes a Banach polynomial ideal (see [**4**, Proposition 3.7]).

By $\widehat{\otimes}_{\pi}^{m,s} E$ and $\widehat{\otimes}_{\pi_s}^{m,s} E$ we denote the m-fold completed symmetric tensor product of E endowed with the projective norm π and the projective s-tensor norm π_s, respectively. The projective norm π is well-known (see e.g. [**16**]) and the projective s-tensor norm π_s is defined by

$$\pi_s(z) = \inf \left\{ \sum_{j=1}^{k} |\lambda_j| \|x_j\|^m : k \in \mathbb{N}, z = \sum_{j=1}^{k} \lambda_j x_j \otimes \cdots \otimes x_j \right\}$$

for $z \in \otimes^{m,s} E$ (see [**9**]).

Given $P \in \mathcal{P}(^m E; F)$, by $\check{P}$ we mean the continuous symmetric m-linear map associated to the polynomial P, that is, the unique symmetric continuous m-linear

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map $\check{P} \in \mathcal{L}(^m E; F)$ fulfilling $\check{P}(x, \ldots, x) = P(x)$ for all $x \in E$. Also,

$$P_L \colon \widehat{\otimes}_{\pi}^{m,s} E \longrightarrow F, \; P_L(x \otimes \cdots \otimes x) = P(x) \text{ and}$$

$$P^L \colon \widehat{\otimes}_{\pi_s}^{m,s} E \longrightarrow F, \; P^L(x \otimes \cdots \otimes x) = P(x)$$

denote the linearizations of P. If we consider the map $\delta_m : E \longrightarrow \otimes^{m,s} E$ given by $\delta_m(x) = x \otimes \cdots \otimes x$, it is clear that $P = P_L \circ \delta_m = P^L \circ \delta_m$. The map δ_m is continuous when $\otimes^{m,s} E$ is endowed with either π or π_s. It is well known that $\|P^L\| = \|P\|$, $\|P_L\| = \|\check{P}\|$ and that

$$\|P\| \leq \|\check{P}\| \leq c(m, E)\|P\|, \tag{2.2}$$

where $c(m, E)$ denotes the $m-$th polarization constant of E. For the general theory of symmetric tensor products we refer to [**9**].

For $1 \leq p < \infty$, let $[\mathcal{QN}_p, \nu_p^Q]$ denote the ideal of *quasi $p-$nuclear operators*. Recall that a linear operator $T : E \to F$ is said to be quasi $p-$nuclear if $j_F \circ T$ is $p-$nuclear, where $j_F : F \to \ell_\infty(B_{F^*})$ is the natural isometric embedding. It is well known that $T \in \mathcal{QN}_p(E; F)$ if and only if there exists a sequence $(x_n^*)_n$ in $\ell_p(E^*)$ such that

$$\|T(x)\| \leq \|(x_n^*(x))_n\|_p$$

for all $x \in E$. In this case,

$$\nu_p^Q(T) = \inf \|(x_n^*)_n\|_{\ell_p(E^*)}$$

is the associated norm, making $\mathcal{QN}_p$ into a Banach space. Here, the infimum is taken over all sequences $(x_n^*)_n$ in $\ell_p(E^*)$ fulfilling the above inequality.

3. The ideal of $p-$compact homogeneous polynomials

Given the Banach operator ideal $[\mathcal{K}_p, k_p]$, we consider the composition ideal of polynomials $\mathcal{K}_p \circ \mathcal{P}$. An m-homogeneous polynomial $P \in \mathcal{P}(^m E; F)$ belongs to $\mathcal{K}_p \circ \mathcal{P}(^m E; F)$ if there are a Banach space G, an m-homogeneous polynomial $Q \in \mathcal{P}(^m E; G)$ and an operator $u \in \mathcal{K}_p(G; F)$ such that $P = u \circ Q$. The composition norm, as in (2.1) is given by

$$k_p(P) := \inf\{k_p(u)\|Q\| : P = u \circ Q, Q \in \mathcal{P}(^m E; G), u \in \mathcal{K}_p(G; F)\},$$

for $P \in \mathcal{K}_p \circ \mathcal{P}(^m E; F)$. If we now consider the space of all continuous $m-$multilinear mappings $\mathcal{L}(^m E; F)$, the composition ideal of multilinear mappings $\mathcal{K}_p \circ \mathcal{L}$ can be defined in a similar way.

We now obtain some characterizations of the ideal $\mathcal{P}_{\mathcal{K}_p}$, which was defined in Section 2. Among other things we show that it is indeed a composition ideal. In fact, in (1) $\iff$ (2) below, we show that $\mathcal{P}_{\mathcal{K}_p} = \mathcal{K}_p \circ \mathcal{P}$.

Theorem 3.1. *Let E and F be Banach spaces. The following are equivalent for an $m-$homogeneous polynomial $P : E \longrightarrow F$:*

(1) $P \in \mathcal{P}_{\mathcal{K}_p}(^m E; F)$.
(2) $P \in \mathcal{K}_p \circ \mathcal{P}(^m E; F)$.
(3) $P_L \in \mathcal{K}_p(\widehat{\otimes}_{\pi}^{m,s} E; F)$.
(4) $P^L \in \mathcal{K}_p(\widehat{\otimes}_{\pi_s}^{m,s} E; F)$.
(5) $\check{P} \in \mathcal{K}_p \circ \mathcal{L}(^m E; F)$.

Moreover,

$$k_p(P) = k_p(P^L) = \inf\{\|(x_n)_n\|_{\ell_p(F)} : P(B_E) \subset p - conv\{(x_n)_n\}.$$

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Proof. Since continuous $m-$homogeneous polynomials map bounded sets to bounded sets, (2)⇒(1) is clear.

To prove (1)⇒(4) take $P \in \mathcal{P}_{\mathcal{K}_p}({}^m E; F)$. Since $B_{\widehat{\otimes}^{n,s}_{\pi_s} E} = \overline{\Gamma}(\delta_m(B_E))$ it follows that

$$P^L(\overline{\Gamma}(\delta_m(B_E))) \subset \overline{\Gamma}(P^L(\delta_m(B_E))) = \overline{\Gamma}(P(B_E)).$$

Part (4) now follows from the fact that the closed absolutely convex hull of a relatively $p-$compact set is $p-$compact.

All other implications and equality of norms follow from [**4**, Propositions 3.2 and 3.7]. □

By the ideal property, the composition of a $p-$compact homogeneous polynomial with a continuous linear operator remains $p-$compact. Let us show a stronger property: the composition of a $p-$compact $m-$homogeneous polynomial with any $n-$homogeneous polynomial is $p-$compact.

Theorem 3.2. *Let $1 \leq p \leq \infty$. Any continuous homogeneous polynomial maps relatively $p-$compact sets to relatively $p-$compact sets.*

Proof. Let $P \in \mathcal{P}({}^m E; F)$. Since $P = P_L \circ \delta_m$ and P_L is continuous and linear, it suffices to prove that δ_m maps relatively $p-$compact sets to relatively $p-$compact sets.

Let $(x_n)_n \in \ell_p(E)$. Given $x \in p-\mathrm{conv}\{(x_n)_n\}$, there exists a sequence $(a_n)_n \in B_{\ell_{p'}}$ such that $x = \sum_{n=1}^{\infty} a_n x_n$. Then a calculation shows that

$$\begin{aligned}
\delta_m(x) &= \delta_m\left(\sum_{n=1}^{\infty} a_n x_n\right) \\
&= \sum_{n=1}^{\infty} a_n x_n \otimes \cdots \otimes \sum_{n=1}^{\infty} a_n x_n \\
&= \sum_{i_1,\ldots,i_m=1}^{\infty} a_{i_1}\cdots a_{i_m} x_{i_1} \otimes \cdots \otimes x_{i_m} \qquad (3.1)\\
&= \sum_{i_1 \leq \cdots \leq i_m} a_{i_1}\cdots a_{i_m} \frac{1}{b_{i_1,\ldots,i_m}} \left(\sum_{\sigma \in S_m} x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(m)}}\right) \qquad (3.2)
\end{aligned}$$

Here $b_{i_1,\ldots,i_m} = k_1!\cdots k_p!$ whenever the vector $(i_1,\ldots,i_m)$ contains p different entries, say $i_{j_1},\ldots,i_{j_p}$ and each i_{j_l} appears k_l times in $(i_1,\ldots,i_m)$. Notice that $k_1 + \cdots + k_p = m$. Indeed, in this case there are $\frac{m!}{k_1!\cdots k_p!}$ summands in (3.1) subindexed with the coordinates of $(i_1,\ldots,i_m)$, whereas there are $m!$ summands in (3.2). Therefore, it is easy to conclude that $b_{i_1,\ldots,i_m} = k_1!\cdots k_p!$

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Notice that each $\sum_{\sigma\in S_m} x_{i_{\sigma(1)}}\otimes\cdots\otimes x_{i_{\sigma(m)}}$ is a symmetric tensor. Moreover,

$$\begin{aligned}
&\sum_{i_1\le\cdots\le i_m}\left[\pi\left(\frac{1}{b_{i_1,\dots,i_m}}\sum_{\sigma\in S_m} x_{i_{\sigma(1)}}\otimes\cdots\otimes x_{i_{\sigma(m)}}\right)\right]^p\\
&\le \sum_{i_1\le\cdots\le i_m}\frac{1}{b^p_{i_1,\dots,i_m}}(m!)^p\|x_{i_1}\|^p\cdots\|x_{i_m}\|^p\\
&\le (m!)^p\sum_{i_1=1}^{\infty}\|x_{i_1}\|^p\left(\sum_{i_2=i_1}^{\infty}\|x_{i_2}\|^p\cdots\left(\sum_{i_m=i_{m-1}}^{\infty}\|x_{i_m}\|^p\right)\cdots\right)\\
&\le (m!)^p\|(x_n)_n\|_p^{mp}<\infty
\end{aligned}$$

Therefore the sequence $(y_n)_n := \left(\frac{1}{b_{i_1,\dots,i_m}}\sum_{\sigma\in S_m} x_{i_{\sigma(1)}}\otimes\cdots\otimes x_{i_{\sigma(m)}}\right)_{i_1\le\dots\le i_m}$ belongs to $\ell_p(\widehat{\otimes}^{m,s}_\pi E)$.

On the other hand,

$$\begin{aligned}
\sum_{i_1\le\cdots\le i_m}|a_{i_1}\cdots a_{i_m}|^{p'} &= \sum_{i_1=1}^{\infty}|a_{i_1}|^{p'}\left(\sum_{i_2=i_1}^{\infty}|a_{i_2}|^{p'}\cdots\left(\sum_{i_m=i_{m-1}}^{\infty}|a_{i_m}|^{p'}\right)\cdots\right)\\
&\le \|(a_n)_n\|_{p'}^{mp'}\le 1
\end{aligned}$$

Then $\delta_m(x)\in p-\mathrm{conv}\{(y_n)_n\}$ and so $\delta_m(p-\mathrm{conv}\{(x_n)_n\})$ is relatively $p-$compact. □

Corollary 3.3. Let E, F and G be Banach spaces and let $1\le p\le\infty$. If $P\in\mathcal{P}_{\mathcal{K}_p}(^mE;G)$ and $Q\in\mathcal{P}(^nG;F)$ then $Q\circ P\in\mathcal{P}_{\mathcal{K}_p}(^{mn}E;F)$. Moreover,

$$k_p(Q\circ P)\le\|Q_L\|k_p(P)^n\le c(n,G)\|Q\|k_p(P)^n.$$

Proof. The first assertion is an easy consequence of Theorem 3.2. The second inequality follows from (2.2). Let us prove that, under the assumptions,

$$k_p(Q\circ P)\le\|Q_L\|k_p(P)^n.$$

Let $\epsilon>0$. Let $(x_j)_j\in\ell_p(G)$ be such that $P(B_E)\subset p-\mathrm{conv}\{(x_j)_j\}$ and $k_p(P)+\epsilon\ge\|(x_j)_j\|_p$. Following the proof and notation of Theorem 3.2 we have that

$$\begin{aligned}
Q(P(B_E)) &\subset Q_L\circ\delta_n(p-conv\{(x_j)_j\})\\
&\subset Q_L\big(p-conv\big\{\big(\frac{1}{b_{i_1,\dots,i_m}}\sum_{\sigma\in S_m}x_{i_{\sigma(1)}}\otimes\cdots\otimes x_{i_{\sigma(m)}}\big)_{i_1\le\dots\le i_m}\big\}\big)\\
&= p-conv\big\{\big(Q_L\big(\frac{1}{b_{i_1,\dots,i_m}}\sum_{\sigma\in S_m}x_{i_{\sigma(1)}}\otimes\cdots\otimes x_{i_{\sigma(m)}}\big)\big)_{i_1\le\dots\le i_m}\big\}.
\end{aligned}$$

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Then,

$$
\begin{aligned}
k_p(Q\circ P) &\le \Big\|\big(Q_L\big(\frac{1}{b_{i_1,\dots,i_m}}\sum_{\sigma\in S_m} x_{i_{\sigma(1)}}\otimes\cdots\otimes x_{i_{\sigma(m)}}\big)\big)_{i_1\le\dots\le i_m}\Big\|_p \\
&= \left(\sum_{i_1\le\dots\le i_n}\big\|Q_L\big(\frac{1}{b_{i_1,\dots,i_m}}\sum_{\sigma\in S_m} x_{i_{\sigma(1)}}\otimes\cdots\otimes x_{i_{\sigma(m)}}\big)\big\|^p\right)^{1/p} \\
&\le \|Q_L\|\left(\sum_{i_1\le\dots\le i_n}\pi\big(\frac{1}{b_{i_1,\dots,i_m}}\sum_{\sigma\in S_m} x_{i_{\sigma(1)}}\otimes\cdots\otimes x_{i_{\sigma(m)}}\big)^p\right)^{1/p} \\
&\le \|Q_L\|\big(\sum_{i_1\le\dots\le i_n}\frac{1}{b_{i_1,\dots,i_m}}\sum_{\sigma\in S_m}\|x_{i_{\sigma(1)}}\|^p\cdots\|x_{i_{\sigma(m)}}\|^p\big)^{1/p} \\
&= \|Q_L\|\big(\sum_{i_1,\dots,i_n=1}^{\infty}\|x_{i_1}\|^p\cdots\|x_{i_m}\|^p\big)^{1/p} \\
&= \|Q_L\|\big(\sum_{i_1=1}^{\infty}\|x_{i_1}\|^p\big)^{1/p}\cdots\big(\sum_{i_n=1}^{\infty}\|x_{i_n}\|^p\big)^{1/p} \\
&= \|Q_L\|\|(x_i)_i\|_p^n \\
&\le \|Q_L\|(k_p(P)+\epsilon)^n.
\end{aligned}
$$

As ϵ is arbitrary the conclusion follows. □

Let us exploit once more the ideas and calculations used in the proof of Theorem 3.2. They now allow us to get the stability of the ideal $\mathcal{K}_p$ under the formation of symmetric tensor products. If $T: E\to F$ is a continuous linear operator, $\otimes_m T$ denotes the continuous linear operator $\otimes_m T:\hat{\otimes}_{\pi_s}^{m,s}E\to\hat{\otimes}_{\pi}^{m,s}F$ given by $\otimes_m T(\sum_{i=1}^n \alpha_i x_i\otimes\cdots\otimes x_i)=\sum_{i=1}^n\alpha_i T(x_i)\otimes\cdots\otimes T(x_i)$, which is then extended by continuity to the completions.

Theorem 3.4. *Let $1\le p<\infty$ and let E and F be Banach spaces. If $T\in\mathcal{K}_p(E;F)$ then $\otimes_m T\in\mathcal{K}_p(\hat{\otimes}_{\pi_s}^{m,s}E;\hat{\otimes}_{\pi}^{m,s}F)$ for every $m\in\mathbb{N}$.*

Proof. By assumption there exists a sequence $(x_n)_n\in\ell_p(F)$ such that

$$T(B_E)\subset p-conv\{(x_n)_n\}. \tag{3.3}$$

Since $B_{\hat{\otimes}_{\pi_s}^{m,s}E}=\overline{\Gamma}(\delta_m(B_E))$, the linear map $\otimes_m T$ is continuous and the closed absolutely convex hull of a relatively $p-$compact set is $p-$compact, it suffices to prove that $\otimes_m T(\delta_m(B_E))$ is relatively $p-$compact.

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Let $x \in B_E$. By (3.3) we can write $T(x) = \sum_{i=1}^{\infty} a_i x_i$, for some sequence $(a_i)_i \in B_{\ell_{p'}}$. Then,

$$\begin{aligned}
\otimes_m T(\delta_m(B_E)) &= T(x) \otimes \cdots \otimes T(x) \\
&= \sum_{i=1}^{\infty} a_i x_i \otimes \cdots \otimes \sum_{i=1}^{\infty} a_i x_i \\
&= \sum_{i_1,\dots,i_m=1}^{\infty} a_{i_1} \cdots a_{i_m} x_{i_1} \otimes \cdots \otimes x_{i_m} \\
&= \sum_{i_1 \le \cdots \le i_m} a_{i_1} \cdots a_{i_m} \frac{1}{b_{i_1,\dots,i_m}} \sum_{\sigma \in S_m} x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(m)}},
\end{aligned}$$

where $b_{i_1,\dots,i_m}$ are as in the proof of Theorem 3.2. The same calculations show that the sequence

$$(y_n)_n := \left(\frac{1}{b_{i_1,\dots,i_m}} \sum_{\sigma \in S_m} x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(m)}} \right)_{i_1 \le \dots \le i_m}$$

belongs to $\ell_p(\widehat{\otimes}_{\pi}^{m,s} F)$ and that $(a_{i_1} \cdots a_{i_m})_{i_1 \le \cdots \le i_m}$ is in $B_{\ell_{p'}}$. Then

$$\otimes_m T(\delta_m(B_E)) \in p-conv\{(y_n)_n\}.$$

Hence, $\otimes_m T$ is $p-$compact. □

The lack of associativity in the projective symmetric tensor product does not permit us to define the tensor product $\otimes_n P$ of an $m-$homogeneous polynomials P for $n \neq m$. However, the next definition shows how to handle the case $n = m$. Let $P \in \mathcal{P}(^m E; F)$. The $m-$*tensor product* of P is the $m-$homogeneous polynomial $\otimes_m P \in \mathcal{P}(^m \hat{\otimes}_{\pi_s}^{m,s} E; \hat{\otimes}_{\pi}^{m,s} F)$ given by $\otimes_m P = (\otimes_m P^L) \circ \delta_m$.

The commutativity of the diagram

$$\begin{tikzcd}
\hat{\otimes}_{\pi_s}^{m,s} E \arrow[rr, "\otimes_m P"] \arrow[dr, "\delta_m"'] & & \hat{\otimes}_{\pi}^{m,s} F \\
& \hat{\otimes}_{\pi_s}^{m,s}(\hat{\otimes}_{\pi_s}^{m,s} E) \arrow[ur, "\otimes_m P^L"'] &
\end{tikzcd}$$

makes clear that $(\otimes_m P)^L = \otimes_m P^L$. Notice that $\hat{\otimes}_{\pi_s}^{m,s}(\hat{\otimes}_{\pi_s}^{m,s} E)$ and $\hat{\otimes}_{\pi_s}^{m^2,s} E$ may differ. Then, although $\mathcal{P}(^m \hat{\otimes}_{\pi_s}^{m,s} E; \hat{\otimes}_{\pi}^{m,s} F)$ and $\mathcal{L}(\hat{\otimes}_{\pi_s}^{m,s}(\hat{\otimes}_{\pi_s}^{m,s} E); \hat{\otimes}_{\pi}^{m,s} F)$ are isometrically isomorphic via the canonical linearization, we cannot conclude that $(\otimes_m P)^L$ belongs to $\mathcal{L}(\hat{\otimes}_{\pi_s}^{m^2,s} E; \hat{\otimes}_{\pi}^{m,s} F)$.

The composition ideal $\mathcal{I} \circ \mathcal{P}$ is *stable* under the formation of symmetric tensor products if $\otimes_m P$ belongs to $\mathcal{I} \circ \mathcal{P}(^m \hat{\otimes}_{\pi_s}^{m,s} E; \hat{\otimes}_{\pi}^{m,s} F)$ for all $P \in \mathcal{I} \circ \mathcal{P}(^m E; F)$.

The next result shows that the stability by forming tensor products of an operator ideal can be transferred to the ideal of polynomials obtained by composition.

Proposition 3.5. If an operator ideal $\mathcal{I}$ is stable under the formation of symmetric tensor products then so is the composition ideal of polynomials $\mathcal{I} \circ \mathcal{P}$.

Proof. Let $P \in \mathcal{I} \circ \mathcal{P}(^m E; F)$. By [**4**, Propositions 3.2] $P^L \in \mathcal{I}(\hat{\otimes}_{\pi_s}^{m,s} E; F)$. By the hypothesis and the comments above,

$$(\otimes_m P)^L = \otimes_m P^L \in \mathcal{I}(\hat{\otimes}_{\pi_s}^{m,s}(\hat{\otimes}_{\pi_s}^{m,s} E); \hat{\otimes}_{\pi}^{m,s} F).$$

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Then $\otimes_m P \in \mathcal{I} \circ \mathcal{P}(^m\hat{\otimes}^{m,s}_{\pi_s} E; \hat{\otimes}^{m,s}_{\pi} F)$. $\square$

From Theorem 3.4 and the above proposition, we have the following.

COROLLARY 3.6. If $P \in \mathcal{P}_{\mathcal{K}_p}(^m E; F)$ then $\otimes_m P \in \mathcal{P}_{\mathcal{K}_p}(^m\hat{\otimes}^{m,s}_{\pi_s} E; \hat{\otimes}^{m,s}_{\pi} F)$.

The next result solves a problem related to transposes that appeared in [**2**]. The notion of transpose of a compact operator was extended in [**3**, Proposition 3.2] to the case of an $m-$homogeneous polynomial $P : E \to F$ as follows. For $P \in \mathcal{P}(^m E; F)$, the transpose of P is defined as the continuous linear operator $P^* : F^* \to \mathcal{P}(^m E)$ given by $P^*(\varphi)(x) = \varphi(P(x))$ $(\varphi \in F^*, x \in E)$. Among other things, it was shown that P is compact if and only if P^* is compact. In [**6**] it is proved that an operator $T : E \to F$ is $p-$compact if and only if it transpose $T^* : F^* \to E^*$ is quasi $p-$nuclear. In a similar way to the linear case, we get the analogous result for polynomials in Theorem 3.8 below. In order to establish this result we will make use of the following lemma, whose proof is based on [**6**, Proposition 3.1].

LEMMA 3.7. *Let* $P \in \mathcal{P}(^m E; F)$ *and* $1 \leq p < \infty$. *Given* $(y_n)_n \in \ell_p^w(F)$, $P(B_E) \subset \overline{p - conv}\{(y_n)_n\}$ *if and only if* $\|P^*(y^*)\| \leq \|(y^*(y_n))_n\|_p$ *for all* $y^* \in F^*$.

PROOF. Assume first that $\|P^*(y^*)\| \leq \|(y^*(y_n))_n\|_p$ for all $y^* \in F^*$, but there is $x_0 \in B_E$ such that $P(x_0)$ does not belong to $\overline{p - conv}(y_n)$. As $p-$conv$\{(y_n)_n\}$ is absolutely convex, by the Hahn-Banach theorem there is $y^* \in F^*$ and $\alpha > 0$ such that $|y^*(P(x_0))| > \alpha$ and $|y^*(y)| \leq \alpha$ for all $y \in p-$conv$\{(y_n)_n\}$. Then

$$\begin{aligned} \alpha &< |y^*(P(x_0))| = |P^*(y^*)(x_0)| \\ &\leq \|P^*(y^*)\| \|x_0\|^m \\ &\leq \|P^*(y^*)\| \\ &\leq \|(y^*(y_n))_n\|_p \\ &\leq \alpha, \end{aligned}$$

a contradiction.

Assume now that $P(B_E) \subset \overline{p - conv}\{(y_n)_n\}$. Given $\epsilon > 0$ and $y^* \in B_{F^*}$ choose $x \in B_E$ such that

$$\|P^*(y^*)\| - \frac{\epsilon}{2} < |P^*(y^*)(x)| = |y^*(P(x))|.$$

Take $(\alpha_n)_n \in B_{\ell_{p'}}$ with

$$\|P(x) - \sum_{n=1}^{\infty} \alpha_n y_n\| \leq \frac{\epsilon}{2}. \tag{3.4}$$

Then,

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$$\begin{aligned}
\|P^*(y^*)\| &\leq |y^*(P(x))| + \frac{\epsilon}{2} \\
&\leq |y^*(P(x) - \sum_{n=1}^{\infty} \alpha_n y_n)| + |y^*(\sum_{n=1}^{\infty} \alpha_n y_n)| + \frac{\epsilon}{2} \\
&\leq \frac{\epsilon}{2} + \sum_{n=1}^{\infty} |\alpha_n||y^*(y_n)| + \frac{\epsilon}{2} \\
&\leq \|(\alpha_n)_n\|_{p'} \|(y^*(y_n))_n\|_p + \epsilon \\
&\leq \|(y^*(y_n))_n\|_p + \epsilon
\end{aligned}$$

Since ϵ is arbitrary the result follows. □

We observe that if $p > 1$, then in fact $\overline{p - conv}\{(y_n)_n\} = p - conv\{(y_n)_n\}$. Hence, the argument above is easier since (3.4) can be replaced by $P(x) = \sum_{n=1}^{\infty} \alpha_n y_n$.

Theorem 3.8. *Let $P \in \mathcal{P}(^m E; F)$ and $1 \leq p < \infty$. Then P is $p-$compact if and only if P^* is quasi $p-$nuclear. In this case, $\nu_p^Q(P^*) \leq k_p(P)$.*

Proof. Assume first that P is $p-$compact. Given $\epsilon > 0$, choose $(y_n)_n \in \ell_p(F)$ such that $P(B_E) \subset p-\text{conv}\{(y_n)_n\}$ and $k_p(P) + \epsilon > \|(y_n)_n\|_p$. By Lemma 3.7,

$$\|P^*(y^*)\| \leq \|(y^*(y_n))_n\|_p$$

for all $y^* \in F^*$. Then $P^* \in \mathcal{QN}_p(F^*, \mathcal{P}(^m E))$ and

$$\nu_p^Q(P^*) \leq \|(y_n)_n\|_p < k_p(P) + \epsilon.$$

Conversely, if $P^* \in \mathcal{QN}_p(F^*; \mathcal{P}(^m E))$, by [**6**, Corollary 3.4] it follows that $P^{**} \in \mathcal{K}_p(\mathcal{P}(^m E)^*, F^{**})$. Consider the evaluation map $\delta : E \to \mathcal{P}(^m E)^*$ given by $\delta_x(P) = P(x)$. Since δ maps B_E into $B_{\mathcal{P}(^m E)^*}$, it follows that $P^{**} \circ \delta(B_E)$ is relatively $p-$compact. On the other hand, $P^{**} \circ \delta = j_F \circ P$, where $j_F : F \to F^{**}$ is the natural injection. Then, by [**6**, Corollary 3.6], $P(B_E)$ is relatively $p-$compact in F. □

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