BOUNDARIES FOR SPACES OF HOLOMORPHIC FUNCTIONS ON M-IDEALS IN THEIR BIDUALS

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Abstract. For a complex Banach space X, let $\mathcal{A}_u(B_X)$ be the Banach algebra of all complex valued functions defined on B_X that are uniformly continuous on B_X and holomorphic on the interior of B_X, and let $\mathcal{A}_{wu}(B_X)$ be the Banach subalgebra consisting of those functions in $\mathcal{A}_u(B_X)$ that are uniformly weakly continuous on B_X. In this paper we study a generalization of the notion of *boundary* for these algebras, originally introduced by Globevnik. In particular, we characterize the boundaries of $\mathcal{A}_{wu}(B_X)$ when the dual of X is separable. We exhibit some natural examples of Banach spaces where this characterization provides concrete criteria for the boundary. We also show that every non-reflexive Banach space X which is an M-ideal in its bidual cannot have a minimal closed boundary for $\mathcal{A}_u(B_X)$.

1. Introduction

A classical result of Šilov ([30], [31, Theorem 7.4] or [17, Theorem I.4.2]) states that if K is a compact Hausdorff topological space and $\mathcal{A}$ is a unital and separating subalgebra of $\mathcal{C}(K)$ then there is a minimal closed subset $M \subset K$ such that $\|f\| = \max_{m \in M} |f(m)|$ for every $f \in \mathcal{A}$. This set is known as the **Šilov** boundary for $\mathcal{A}$.

Five years after Šilov's paper, Bishop ([11, Theorem 1]) proved that if $\mathcal{A}$ is a separating Banach algebra of continuous functions on a compact metrizable space K, then K has a minimal subset M (not necessarily closed) satisfying the following condition:

$$\text{For all } f \in \mathcal{A}, \text{ there exists } m \in M \text{ such that } |f(m)| = \|f\| \qquad (*) \ .$$

In fact, M is the set of all peak points (see definition below) for $\mathcal{A}$.

About twenty years after the classical results of Šilov and Bishop, Globevnik [21] considered the problem of extending those results to the setting of certain subalgebras $\mathcal{A} \subset \mathcal{C}_b(\Omega)$, where Ω is a topological space which is not necessarily compact, and where $\mathcal{C}_b(\Omega)$ denotes the space of continuous and bounded functions on Ω endowed with the usual sup norm.

Following Globevnik, we make the following definition.

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Definition 1.1. We say that a subset Γ of a topological space Ω is a **boundary** for a function algebra $\mathcal{A} \subset \mathcal{C}_b(\Omega)$ if

$$\|f\| = \sup\{|f(x)| : x \in \Gamma\}, \quad \text{for all } f \in \mathcal{A}.$$

Globevnik studied the boundaries for two important algebras of holomorphic functions that are closed subalgebras of $\mathcal{C}_b(\Omega)$ in case Ω is the closed unit ball of the complex Banach space c_0 (see [20, 21]). The pioneering work of Globevnik was followed by many authors (see, e.g., [6], [27], [28], [13], [1], [2], [14], [4], [15] and [12]), who studied the existence and characterization of such generalized boundaries. These authors considered algebras of holomorphic mappings defined on the closed unit ball of concrete complex Banach spaces which were, by and large, sequence spaces.

Our aim here is to study more general situations, and our results will contain many of the cases already studied. We will be more specific below, after introducing some notation and definitions.

Some classes of subsets of Ω play an essential role in the characterization of the boundaries for the subalgebras of $\mathcal{C}_b(\Omega)$ that we are going to consider. We introduce these classes of sets in a more general setting, since they will be important in the characterization of the boundaries for other algebras of functions.

If Ω is a topological space, an element $x_0 \in \Omega$ is a **peak point** for a subspace $\mathcal{A}$ of $\mathcal{C}_b(\Omega)$ if there exists a function $f \in \mathcal{A}$ such that

$$f(x_0) = 1 \quad \text{and} \quad |f(x)| < 1, \quad \text{for all } x \in \Omega \backslash \{x_0\} .$$

If Ω is a metric space, then $x_0 \in \Omega$ is a **strong peak point** for $\mathcal{A}$ if there exists a function $f \in \mathcal{A}$ such that $f(x_0) = 1$ and for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } d(x, x_0) > \varepsilon, \quad \text{then} \quad |f(x)| < 1 - \delta .$$

Throughout, X will be a complex Banach space with unit sphere, resp. closed unit ball, denoted by S_X, resp. B_X. The dual of X will be written X'. A point $x \in S_X$ is called a **complex extreme point** of B_X if

$$y \in X, \quad \|x + \lambda y\| \leq 1 \text{ for all } \lambda \in \mathbb{C}, |\lambda| \leq 1 \quad \text{implies that} \quad y = 0 .$$

Given a Banach space X, the set of all complex extreme points of B_X will be denoted by $\mathrm{Ext}_{\mathbb{C}} B_X$.

If $\mathcal{A}$ is a subalgebra of $\mathcal{C}_b(\Omega)$ containing 1 and $\mathcal{S}_1^* = \{\varphi \in \mathcal{A}' : \varphi(1) = 1 = \|\varphi\|\}$, then to each $x \in \Omega$ corresponds the element $\delta_x \in \mathcal{S}_1^*$, where $\delta_x(f) = f(x)$ for all $f \in \mathcal{A}$. The **Choquet boundary** for $\mathcal{A}$ is the set $\chi\mathcal{A}$ of all $x \in \Omega$ such that δ_x is an extreme point of $\mathcal{S}_1^*$. Given a Banach subalgebra $\mathcal{A}$ of $\mathcal{C}_b(\Omega)$, we denote by $\rho\,\mathcal{A}$ and by $\partial\,\mathcal{A}$, respectively, the set of peak points for $\mathcal{A}$ and the closed boundary contained in every closed boundary for $\mathcal{A}$. We remark that if Ω is a compact Hausdorff space, $\partial\,\mathcal{A}$ exists and is called the **Šilov boundary** for $\mathcal{A}$. However, without the presence of compactness, $\partial\,\mathcal{A}$ may fail to exist.

In Section 2, we study boundaries for the algebra $\mathcal{A}_u(B_X)$ (resp. $\mathcal{A}_{wu}(B_X)$) of uniformly (resp. weakly uniformly) continuous functions on B_X that are holomorphic on the interior of the ball. We give special attention to the case when X is the canonical predual of a Lorentz sequence space, characterizing the complex extreme points of the unit ball of the bidual of this space, and thereby obtaining precise information about boundaries for $\mathcal{A}_{wu}(B_X)$.

In the final section, we provide a wide class of Banach spaces X for which there is no minimal boundary for the algebra $\mathcal{A}_\infty(B_X)$ of continuous and bounded functions on B_X that are holomorphic on the interior of B_X. In particular, we show that this occurs when X is a proper M-ideal in X'' and, after renorming, when X contains a complemented copy of c_0.

2. Boundaries for some algebras of holomorphic functions

Various types of boundaries have already been considered in [6] and [12]. For instance, the set T of complex extreme points of B_{ℓ_∞} is a closed boundary in the sense of Definition 1.1 for the larger algebra $\mathcal{A}_\infty(B_{\ell_\infty})$, but there are also closed boundaries $\Gamma \subset B_{\ell_\infty}$ such that $\Gamma \cap T = \varnothing$. On the other hand, S_{ℓ_p} is the intersection of all closed boundaries for $\mathcal{A}_\infty(B_{\ell_p})$, $1 \leq p < \infty$. In [12], the authors describe boundaries for $\mathcal{A}_u(B_X)$ for certain spaces X whose bidual is a Marcinkiewicz space. They show that the set of complex extreme points of B_X is a non-empty closed set but that no minimal boundary for $\mathcal{A}_u(B_X)$ exists.

For a complex Banach space X, let $\mathcal{A}_{wu}(B_X)$ be the Banach algebra of those complex valued functions defined on the closed unit ball B_X of X that are weakly uniformly continuous on B_X and holomorphic on the interior of B_X, endowed with the sup norm and let $\mathcal{A}_{w^*u}(B_{X''})$ be the Banach algebra of all complex valued functions defined on the closed unit ball $B_{X''}$ of X'' that are weakly star (uniformly) continuous on $B_{X''}$ and holomorphic on the interior of $B_{X''}$, endowed with the sup norm. In this section we are going to study the boundaries for the Banach algebra $\mathcal{A}_{wu}(B_X)$ in case X is a complex Banach space whose dual is separable. We will present a characterization of the boundaries for $\mathcal{A}_{wu}(B_X)$ in terms of the complex extreme points of $B_{X''}$. Later we will apply the general results to some special cases.

The space of all holomorphic functions from X into $\mathbb{C}$ that, when restricted to any bounded subset of X, are uniformly weakly continuous will be denoted by $H_{wu}(X)$, and the space of all holomorphic functions from X'' into $\mathbb{C}$ that, when restricted to any bounded subset of X'', are (uniformly) w^*-continuous will be denoted by $H_{w^*u}(X'')$. For every non-negative integer n, we will write $\mathcal{P}_{wu}(^nX) := \mathcal{P}(^nX) \cap H_{wu}(X)$ and $\mathcal{P}_{w^*u}(^nX'') := \mathcal{P}(^nX'') \cap H_{w^*u}(X'')$, where $\mathcal{P}(^nX)$ is the space of continuous n-homogeneous polynomials on X. Let $\mathcal{A}_u(B_X)$ denote the Banach algebra of all complex valued functions defined on B_X which are uniformly continuous on B_X and holomorphic on the interior of B_X endowed with the sup norm. Let $H_b(X)$ be the space of all holomorphic functions from X into $\mathbb{C}$ that are bounded on bounded subsets of X. Since $(B_{X''}, w^*)$ is compact, the boundaries Γ for $\mathcal{A}_{w^*u}(B_{X''})$ (in the sense of Globevnik) that are w^*-closed are boundaries for $\mathcal{A}_{w^*u}(B_{X''})$ in the standard sense, that each $f \in \mathcal{A}_{w^*u}(B_{X''})$ achieves its norm.

Proposition 2.1. *The space of all functions $f \in \mathcal{A}_{wu}(B_X)$ such that $f = g|B_X$ for some $g \in H_{wu}(X)$ is dense in $\mathcal{A}_{wu}(B_X)$. Moreover, for each $f \in \mathcal{A}_{wu}(B_X)$ there exists a unique*

$\tilde{f} \in \mathcal{A}_{w^*u}(B_{X''})$ such that $\tilde{f}|B_X = f$. In addition, the extension mapping $f \to \tilde{f}$ is an algebraic and isometric isomorphism.

Proof. The proof of the density in $\mathcal{A}_{wu}(B_X)$ of the space of all functions $f \in \mathcal{A}_{wu}(B_X)$ such that $f = g|B_X$ for some $g \in H_{wu}(X)$ can be found in [18, Proposition 6.1]. Given $f \in \mathcal{A}_{wu}(B_X)$, since f is uniformly w-continuous on B_X, B_X is w*-dense in $B_{X''}$ and $\sigma(X'', X')_{|X} = \sigma(X, X')$, there exists a unique extension $\tilde{f}$ of f to $B_{X''}$ which is w^*-continuous. By the above, there exists a sequence (g_n) in $H_{wu}(X)$ such that $(g_{n|B_X})$ converges to f in $\mathcal{A}_{wu}(B_X)$. For each n, take the unique $\tilde{g}_n \in H_{w^*u}(X'')$ that extends g_n ([26, Theorem 8 and Remark 9]) . It is easy to check that $(\tilde{g}_n|B_{X''})$ is a Cauchy net in the Banach algebra $\mathcal{A}_{w^*u}(B_{X''})$, and so it converges to a function $g \in \mathcal{A}_{w^*u}(B_{X''})$ that satisfies $g = \tilde{f}$. The equality $\| \tilde{f} \| = \| f \|$ follows by density, as does the fact that the extension is a homomorphism. This completes the proof. □

Remark 2.2. The above proposition shows that the Banach algebra $\mathcal{A}_{wu}(B_X)$ enjoys the property that it is "essentially" the uniform algebra $\mathcal{A}_{w^*u}(B_{X''})$. As such, questions about boundaries for the algebra $\mathcal{A}_{wu}(B_X)$ transfer to similar, easier questions for $\mathcal{A}_{w^*u}(B_{X''})$. Thus, for instance, a subset $\Gamma \subset B_X$ is a boundary for $\mathcal{A}_{wu}(B_X)$ (in the sense of Definition 1.1) if and only if Γ is weak-star dense in the Šilov boundary $\partial\mathcal{A}_{w^*u}(B_{X''})$.

Roughly speaking, this occurs because $\mathcal{A}_{wu}(B_X)$ consists of a relatively small set of functions that have very good properties. On the other hand, except in certain very unusual situations, the algebras $\mathcal{A}_u(B_X)$ and $\mathcal{A}_\infty(B_X)$ are much larger and have a much more complicated maximal ideal space (homomorphism) structure. Thus, for instance, one can find a copy of $\beta\mathbb{N}\backslash\mathbb{N}$ in the fiber structure over interior points of $\mathcal{A}_u(B_{\ell_2})$ [7], and so the Šilov boundary of such algebras seems extremely difficult to determine.

Proposition 2.3. *If X is a Banach space whose dual X' is separable and Γ is a subset of B_X, then Γ is a boundary for $\mathcal{A}_{wu}(B_X)$ if and only if $\overline{\Gamma}^{w^*}$ contains all the complex extreme points of $B_{X''}$.*

Proof. First, we recall that $(B_{X''}, w^*)$ is a compact Hausdorff topological space which is also metrizable since X' is separable. Let $P(B_{X''}) \subset \mathcal{C}(B_{X''})$ denote the Banach algebra generated by the constants and the restrictions to $B_{X''}$ of the elements of X'. So, by using a result due to Arenson [5], we have that $\mathrm{Ext}_{\mathbb{C}} B_{X''} = \chi P(B_{X''})$ and in view of [25, Theorem 9.7.2], we have $\chi\mathcal{P}(B_{X''}) = \rho\mathcal{P}(B_{X''})$. It is clear that $\rho\mathcal{P}(B_{X''}) \subset \rho\mathcal{A}_{w^*u}(B_{X''})$ and by [20, Theorem 4] we know that $\rho\mathcal{A}_{w^*u}(B_{X''}) \subset \mathrm{Ext}_{\mathbb{C}} B_{X''}$. From this we deduce that $\rho\mathcal{A}_{w^*u}(B_{X''}) = \mathrm{Ext}_{\mathbb{C}} B_{X''}$. Moreover, by Bishop's Theorem ([11, Theorem 1]), we know that $\rho\mathcal{A}_{w^*u}(B_{X''})$ is the minimal boundary for $\mathcal{A}_{w^*u}(B_{X''})$.

We are ready to prove the stated result. If Γ is a boundary for $\mathcal{A}_{wu}(B_X)$, then in view of Proposition 2.1 we have that $\overline{\Gamma}^{w^*}$ is a closed boundary for $\mathcal{A}_{w^*u}(B_{X''})$. Thus, it contains the minimal boundary which in this case coincides with $\mathrm{Ext}_{\mathbb{C}} B_{X''}$. Conversely, if $\mathrm{Ext}_{\mathbb{C}} B_{X''} \subset \overline{\Gamma}^{w^*}$, it is clear that $\overline{\Gamma}^{w^*}$ is a boundary for $\mathcal{A}_{w^*u}(B_{X''})$ and, by Proposition 2.1, Γ is also a boundary for $\mathcal{A}_{wu}(B_X)$. □

Remark 2.4. It is not known if the separability of X' is essential for the necessity in this result.

Now we will apply the previous result to some special cases. We sketch a proof of the following fact, which will be used in these applications.

Proposition 2.5. *Let X be a complex Banach space such that $\mathcal{P}_{wu}(^nX) = \mathcal{P}(^nX)$ for every $n \geq 1$. Then $\mathcal{A}_u(B_X) = \mathcal{A}_{wu}(B_X)$.*

Proof. We show that any $f \in \mathcal{A}_u(B_X)$ is weakly uniformly continuous on B_X. For this, let $\varepsilon > 0$ be arbitrary. Since f is uniformly continuous on B_X, we can find $s > 1$ so that $\sup_{\|x\|\leq 1} |f(x) - f(\frac{x}{s})| < \frac{\varepsilon}{2}$. Since $f_s(x) := f(\frac{x}{s})$ is holomorphic on $s\overset{\circ}{B}_X$ it can be approximated uniformly on B_X by its Taylor series $\sum_{n=0}^{m} P_n$. That is, $\sup_{\|x\|\leq 1} |f_s(x) - \sum_{n=0}^{m} P_n(x)| < \frac{\varepsilon}{2}$. The proof is concluded by observing that each P_n is weakly uniformly continuous on B_X and that $\mathcal{A}_{wu}(B_X)$ is complete. □

It is well known that Propositions 2.3 and 2.5 apply to c_0 . We can also apply Proposition 2.3 to some other special cases.

We start by considering the space T^* defined by Tsirelson in [32]. The Tsirelson's space is a reflexive Banach space with Schauder basis. For all $n \geq 1$ every element of $\mathcal{P}(^nT^*)$ is weakly sequentially continuous (see [16, p. 121]) and, since T^* does not contain a copy of ℓ_1, by [22, Theorem 4] we have that $\mathcal{P}_{wu}(^nT^*) = \mathcal{P}(^nT^*)$ for every $n \geq 1$. So, these propositions apply and we deduce that a subset Γ of B_{T^*} is a boundary for $\mathcal{A}_u(B_{T^*})$, if and only if, $\overline{\Gamma}^w$ contains all the complex extreme points of B_{T^*}.

The definition of the Tsirelson-James space T_J^* can be found in [8], where the authors show that the T_J^* satisfies the equation $\mathcal{P}_{wu}(^nT_J^*) = \mathcal{P}(^nT_J^*)$ for every $n \geq 1$ and that T_J^* has a shrinking basis and so its dual is separable. So, Propositions 2.3 and 2.5 apply and we have that a subset Γ of $B_{T_J^*}$ is a boundary for $\mathcal{A}_u(B_{T_J^*})$ if and only if $\overline{\Gamma}^{w^*}$ contains all the complex extreme points of the closed unit ball of the bidual of T_J^*.

The rest of this section is devoted to another application of Proposition 2.3. In order to do so, we need to recall the definition of the Lorentz sequence space $d(w, 1)$ and its canonical pre-dual $d_*(w, 1)$. For more details we refer to [29, 19].

Given a decreasing sequence w of positive real numbers satisfying $w \in c_0\backslash\ell_1$, the complex Lorentz sequence space $d(w, 1)$ is the space of all sequences $x : \mathbb{N} \longrightarrow \mathbb{C}$ such that

$$\sup\Big\{\sum_{n=1}^{\infty} |x(\sigma(n))|w_n \ : \sigma \in \Pi(\mathbb{N})\Big\} < +\infty$$

(where $\Pi(\mathbb{N})$ is the set of permutations on $\mathbb{N}$), endowed with the norm given by

$$\|x\| = \sup\Big\{\sum_{n=1}^{\infty} |x(\sigma(n))|w_n \ : \sigma \in \Pi(\mathbb{N})\Big\}.$$

Moreover, if for each bounded complex sequence z we define

$$\phi_n(z) = \sup_{|J|=n} \Big\{ \Big(\sum_{j=1}^{n} w_j \Big)^{-1} \sum_{j\in J} |z_j| \Big\}$$

(where $|J|$ denotes the cardinality of the set $J \subset \mathbb{N}$), we consider the space of the complex sequences z such that $\lim_{n\to\infty}\phi_n(z) = 0$, endowed with the norm given by

$$\|z\| = \max_{n\in\mathbb{N}} \phi_n(z) \ .$$

This space is denoted by $d_*(w,1)$, and it is a complex Banach space. It is easy to check that $d_*(w,1) \subset c_0$ as a set and $\{e_i\}$ is a normalized Schauder basis in $d_*(w,1)$ (where $e_i = (\delta_{ji})_{j=1}^{\infty}$, for all $i \in \mathbb{N}$). It is known that the space $d_*(w,1)$ is a predual of the Lorentz sequence space $d(w,1)$ (see [29, 19]).

For a sequence $x \in c_0$, we will denote by x^* the decreasing rearrangement of x, determined by the properties that it is a decreasing sequence of non-negative real numbers such that

$$\{x^*_n : n \in \mathbb{N}\} \subset \{|x_n| : n \in \mathbb{N}\} \subset \{x^*_n : n \in \mathbb{N}\} \cup \{0\},$$

and

$$|\{n \in \mathbb{N} : x^*_n = |x_{n_0}|\}| = |\{n \in \mathbb{N} : |x_n| = |x_{n_0}|\}|, \quad \text{for all } n_0 \in \{n \in \mathbb{N} : x_n \neq 0\} \ .$$

It is known ([29, Lemma 8] and [19, Theorem 11]) that the dual $d'(w,1)$ of $d(w,1)$ is the space of complex sequences z such that $\sup_{n\in\mathbb{N}} \phi_n(z) < \infty$, endowed with the norm given by

$$\|z\| = \sup_{n\in\mathbb{N}} \phi_n(z) \quad (z \in d'(w,1)) \ .$$

We remark that $d'(w,1)$ is a subset of c_0 and

$$\|z\| = \sup_{n} \Big\{ (W_n)^{-1} \sum_{k=1}^{n} z^*_k \Big\}, \quad \text{for all } z \in d'(w,1) \ ,$$

where $W_n = \sum\limits_{j=1}^{n} w_j$ for each positive integer n and $W_0 = 0$.

Our first result shows that in case when $X = d_*(w,1)$ where $w \notin \ell_p$ for all $p \in \mathbb{N}$ the study of $\mathcal{A}_u(B_{d_*(w,1)})$ reduces to the study of the algebra $\mathcal{A}_{wu}(B_{d_*(w,1)})$.

Proposition 2.6. *If $w \notin \ell_p$ for all $p \in \mathbb{N}$ then $\mathcal{A}_u(B_{d_*(w,1)}) = \mathcal{A}_{wu}(B_{d_*(w,1)})$.*

Proof. By using Examples III.1.4c, Corollaries III.3.7 and III.3.3 of [23], every nonreflexive subspace of $d_*(w,1)$ contains an isomorphic copy of c_0. From this we infer that $d_*(w,1)$ does not contain an isomorphic copy of ℓ_1. Since $d_*(w,1)$ has the approximation property, by [9, Proposition 2.12] and [10, Proposition 2.7] we have $\mathcal{P}_{wsc}(^n d_*(w,1)) = \mathcal{P}_{wu}(^n d_*(w,1)) = \overline{\mathcal{P}_f(^n d_*(w,1))}$, where $\mathcal{P}_{wsc}(^n d_*(w,1))$ denotes the space of the complex valued weakly sequentially continuous n-homogeneous polynomials on $d_*(w,1)$. Moreover by [24, Theorem 3.2] we have that $\mathcal{P}(^n d_*(w,1)) = \mathcal{P}_{wsc}(^n d_*(w,1))$ for every $n \geq 2$ whenever $w \notin \ell_p$ for all $p \in \mathbb{N}$, and it is clear that $\mathcal{P}(^1 d_*(w,1)) = \mathcal{P}_{wsc}(^1 d_*(w,1)) = \mathcal{P}_{wu}(^1 d_*(w,1))$. Therefore we have that $\mathcal{P}_{wu}(^n d_*(w,1)) = \mathcal{P}(^n d_*(w,1))$ for every $n \geq 1$ and the result follows by using Proposition 2.5. $\square$

Let us observe that the condition $\mathcal{A}_u(B_{d_*(w,1)}) = \mathcal{A}_{wu}(B_{d_*(w,1)})$ clearly implies that $\mathcal{P}(^n d_*(w,1)) = \mathcal{P}_{wu}(^n d_*(w,1))$ for every n. In view of [24, Theorem 3.2], we deduce that $w \notin \ell_p$ for all $p \geq 1$. Hence the above result is a characterization of when $w \in \ell_p$.

Since the dual of $d_*(w,1)$ is separable, Proposition 2.3 can be used to characterize the boundaries for $\mathcal{A}_{wu}(B_{d_*(w,1)})$. This characterization will be given after we describe the set of complex extreme points of the unit ball of its bidual.

Lemma 2.7. *Let $X = d'(w,1)$ and $z = (z_k) \in B_X$ a non-negative, decreasing sequence of real numbers. Then z is a complex extreme point of B_X if and only if*

$$\liminf \left\{ W_n - \sum_{k=1}^{n} z_k \right\} = 0.$$

Proof. Assume first that (z_k) is a decreasing sequence of non-negative real numbers satisfying $\liminf \left\{ W_n - \sum_{k=1}^{n} z_k \right\} = 0$. Let $y = (y_k) \in X$ be such that for all $\lambda \in \mathbb{C}, |\lambda| = 1$,

$$\|z + \lambda y\| \leq 1. \tag{1}$$

For every $n \in \mathbb{N}$, let us denote

$$t_n := |z_n + y_n| + |z_n - y_n| - 2z_n, \;\; Z_n := \sum_{k=1}^{n} z_k, \;\; \text{and} \;\; T_n := \sum_{k=1}^{n} t_k.$$

It is easy to verify that $t_k \geq 0$ for every $k \in \mathbb{N}$, and consequently (T_n) is an increasing sequence. Moreover, for every $n \in \mathbb{N}$ we have

$$\begin{aligned} Z_n &= \frac{1}{2} \sum_{n=1}^{n} 2z_k = \frac{1}{2}\Big(\sum_{n=1}^{n} |z_k + y_k| + \sum_{n=1}^{n} |z_k - y_k| \Big) - \frac{1}{2} T_n \qquad \text{(by (1))} \\ &\leq \frac{1}{2}(W_n + W_n) - \frac{T_n}{2} = W_n - \frac{T_n}{2}. \end{aligned}$$

From the hypothesis $\liminf \{W_n - Z_n\} = 0$ and from the fact that $\frac{T_n}{2} \leq W_n - Z_n$ for every $n \in \mathbb{N}$ we infer that (T_n) has a subsequence (T_{n_k}) that tends to 0. Consequently, as (T_n) is increasing and non-negative, we get that $T_n = 0$ for every $n \in \mathbb{N}$ and hence $t_n = 0$ for every $n \in \mathbb{N}$. We then obtain that

$$2z_n = |(z_n + y_n) + (z_n - y_n)| = |z_n + y_n| + |z_n - y_n|,$$

and from this equation and from the fact that (z_n) is a sequence of positive real numbers we infer that $y_n \in \mathbb{R}$ for every $n \in \mathbb{N}$. Now, if we repeat the same argument replacing t_n by

$$t'_n = |z_n + iy_n| + |z_n - iy_n| - 2z_n,$$

we obtain that $iy_n \in \mathbb{R}$ for every $n \in \mathbb{N}$. Since $y_n \in \mathbb{R}$ and $iy_n \in \mathbb{R}$, it follows that $y_n = 0$ for every $n \in \mathbb{N}$, so $y = 0$. This proves that z is a complex extreme point of B_X in case that z is a non-negative decreasing sequence.

Conversely, if z is a $\mathbb{C}$-extreme point of B_X, we will prove that $\liminf\left\{W_n-\sum\limits_{k=1}^{n} z_k\right\}=0$. In case $z\in c_{00}$ it is easy to see that z is not a complex extreme point. We argue by contradiction. Suppose that $\liminf\left\{W_n-\sum\limits_{k=1}^{n} z_k\right\}\neq 0$. Since $z\in B_X$ and $\liminf\left\{W_n-\sum\limits_{k=1}^{n} z_k\right\}\neq 0$, it follows that $\liminf\left\{W_n-\sum\limits_{k=1}^{n} z_k\right\}>0$. By the assumption, the set $\{n\in\mathbb{N}:\phi_n(z)=1\}$ is finite. Let $N:=\max\{n\in\mathbb{N}:\phi_n(z)=1\}$ if $\{n\in\mathbb{N}:\phi_n(z)=1\}\neq\emptyset$ and $N=1$ otherwise. As $z\notin c_{00}$ and $z\in c_0$, given $k_0>N$ we may choose $k_0<k<l<m$ as follows: k is the smallest natural number bigger than k_0 such that $z_{k_0}>z_k$, l is the smallest natural number bigger than k such that $z_k>z_l$ and m is the smallest natural number bigger than l such that $z_l>z_m$.

We set $\varepsilon_0:=\frac{1}{2}\inf_{n>N}\left\{W_n-\sum_{s=1}^{n} z_s\right\}, \varepsilon:=\min\left\{z_{k_0}-z_k, z_k-z_l, z_l-z_m,\varepsilon_0\right\}$, and $y:=\varepsilon(e_k-e_l)$. We will verify that

$$\|z\pm y\|\leq 1,\quad \|z\pm iy\|\leq 1\ .$$

We first remark that our assumption implies that $\varepsilon_0>0$ and so ε is positive and $y\neq 0$. It is easily checked that the above conditions imply that z is not an extreme point of B_X.

Clearly, $(z+y)_n^*=z_n^*=z_n$ for every $n\leq k_0$. So for every $j\leq k_0$, we have

$$\sum_{s=1}^{j}(z+y)_s^*=\sum_{s=1}^{j} z_s\leq W_j\ ,$$

while for every $k_0<j$,

$$\begin{aligned}
\sum_{s=1}^{j}(z+y)_s^* &= \sum_{s=1}^{k_0} z_s+\sum_{s=k_0+1}^{j}(z+y)_s^*\\
&\leq \sum_{s=1}^{k_0} z_s+\sum_{s=k_0+1}^{j} z_s+\varepsilon\\
&= \sum_{s=1}^{j} z_s+\varepsilon\\
&\leq W_j\ .
\end{aligned}$$

Therefore, $\|z+y\|\leq 1$ and the same argument also gives $\|z-y\|\leq 1$.

Next, it is clear that

$$|(z+iy)_k|=|z_k+i\varepsilon|\leq z_k+\varepsilon,$$

$$|(z+iy)_l|=|z_l-i\varepsilon|\leq z_l+\varepsilon$$

and that for all $n\leq k_0$,

$$(z+iy)_n^*=z_n.$$

By the choice of m, for every $n\geq m$ and $s\leq m$, we have

$$(z+iy)_n^*=z_n,\quad \text{for all } n\geq m$$

and

$$z_m^* = z_m \leq |z_s + iy_s|, \quad \text{for all } s \leq m \ .$$

So, for all $j \leq k_0$,

$$\sum_{s=1}^{j} (z + iy)_s^* = \sum_{s=1}^{j} z_s \leq W_j.$$

For $k_0 \leq j$, by the choice of ε we have, as in the first case,

$$\sum_{s=1}^{j} (z + iy)_s^* \leq 2\varepsilon + \sum_{s=1}^{j} z_s \leq 2\varepsilon_0 + \sum_{s=1}^{j} z_s \leq W_j \ .$$

This shows that $\|z + iy\| \leq 1$. Using a similar argument, we get that $\|z - iy\| \leq 1$, which shows that z is not a $\mathbb{C}$-extreme point of B_X . □

Theorem 2.8. *Let $X = d'(w, 1)$. If $z \in B_X$, then z is a complex extreme point of B_X if and only if* $\liminf \{W_n - \sum_{k=1}^n z_k^*\} = 0$.

Proof. Before starting the proof, we observe that given any infinite set $M \subset \mathbb{N}$ and an arbitrary bijection σ between $\mathbb{N}$ and M, the subspace $E \subset d'(w, 1)$ given by

$$E := \{x \in d'(w, 1) : x_n = 0 \text{ for all } n \in \mathbb{N}\backslash M\} \tag{2}$$

is isometric to $d'(w, 1)$, by the isometry $T : E \longrightarrow d'(w, 1)$ given by

$$T(x) = (x_{\sigma(n)}) \qquad (x \in E). \tag{3}$$

Now assume that $z \in B_X$ and $\liminf_n\{W_n - \sum_{k=1}^n z_k^*\} = 0$. If $z_k \neq 0$ for every k, it follows from Lemma 2.7 that z is a complex extreme point of the unit ball since the set of complex extreme points of the unit ball is invariant under isometries. Otherwise $\operatorname{supp} z \neq \mathbb{N}$ and the assumption $\liminf_n\{W_n - \sum_{k=1}^n z_k^*\} = 0$ implies that $\operatorname{supp} z$ is infinite. Assume that $\|z + \lambda y\| \leq 1$ for some $y \in d'(w, 1)$ and for every $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$. For $M := \operatorname{supp} z$, consider the subspace E and the isometry $T : E \longrightarrow d'(w, 1)$ defined as in (2) and (3), respectively. Let $P_E : d'(w, 1) \longrightarrow E$ be the canonical projection from $d'(w, 1)$ onto E. It is easy to verify that for all $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$,

$$\|P_E(z + \lambda y)\| \leq 1,$$

and so

$$\|TP_E(z + \lambda y)\| \leq 1.$$

Let us write $\overline{z} := TP_E(z)$ and $\overline{y} := TP_E(y)$. We know that

$$\|\overline{z} + \lambda \overline{y}\| \leq 1,$$

for all $\lambda \in \mathbb{C}, |\lambda| \leq 1$. Since $z_{\sigma(n)} \neq 0$ for every n, we can apply the previous argument to $\overline{z}$ to deduce that $\overline{y} = 0$, that is, $y_{\sigma(n)} = 0$ for every n. This means that $y_m = 0$ for every $m \in M$. We will check that $y = 0$. Assume that $y_k \neq 0$ for some $k \in \mathbb{N}\backslash M$. Since $w \in c_0$, there exists an integer N such that $w_m < |y_k|$ for every $m \geq N$. Let us fix $\varepsilon < |y_k| - w_{N+1}$. By the assumption, $\liminf \{W_n - \sum_{k=1}^n z_k^*\} = 0$, and hence for some $n > N$ we have

$$W_n - \sum_{k=1}^n z_k^* < \varepsilon \ .$$

So, there is $J \subset M$ with $|J| = n$ such that $W_n - \sum_{i \in J} |z_i| < \varepsilon$. From this we infer that

$$W_{n+1} = W_n + w_{n+1} < \sum_{i \in J} |z_i| + \varepsilon + w_{n+1} <$$

$$< \sum_{i \in J} |z_i| + |y_k| - w_{N+1} + w_{n+1} \leq$$

$$\sum_{i \in J} |z_i| + |y_k|$$

and so, since $J \subset M$ and $k \notin M$, $\|z + y\| > 1$. Since this is impossible, we have proved that $y = 0$ and that z is a complex extreme point of B_X.

Conversely, if z is a $\mathbb{C}$-extreme point of B_X, we prove that $\liminf \Big\{ W_n - \sum\limits_{k=1}^n z_k^* \Big\} = 0$. In case $z \in c_{00}$ it is easy to see that z is not a complex extreme point. We remark that $|z| := (|z_k|)$ is a complex extreme point of B_X whenever z is a complex extreme point of B_X. Since $z \notin c_{00}$ we have that $M := \operatorname{supp} z$ is infinite. Let E be the subspace defined in (2) and T the isometry given in (3). Hence $P_E(|z|)$ is a complex extreme point of B_E. Consequently $TP_E(|z|)$ is a complex extreme point of B_X. By Lemma 2.7, we infer that $\liminf \{W_n - \sum_{k=1}^n |z_{\sigma(k)}|\} = 0$, where $\sigma : \mathbb{N} \longrightarrow M$ is a mapping such that $|z_{\sigma(n)}| = z^*(n)$ for every n. This completes the proof since the previous condition implies that $\liminf \{W_n - \sum_{k=1}^n z_k^*\} = 0$. □

Proposition 2.9. *If $X = d_*(w, 1)$, we have $B_{X''} = \overline{\operatorname{Ext}_{\mathbb{C}}(B_{X''})}^{w^*}$.*

Proof. Let us fix $x'' \in B_{X''} \setminus \{0\}$ and choose any $n_0 \in \mathbb{N}$ such that $P_{n_0} x'' \neq 0$, where P_{n_0} is the natural projection. Let m be the number of elements in the set $\mathfrak{I} = \{k \leq n_0 : |x_k''| \neq 0\}$. As

$$\lim_k \Big\{ \frac{W_{m+k+1} - W_{m+k}}{k + 1 - k} \Big\} = \lim_k \{w_{m+k+1}\} = 0,$$

so

$$\lim_k \Big\{ \frac{W_{m+k}}{k} \Big\} = 0 \ .$$

Hence

$$\lim_k \Big\{ \frac{1}{k} \Big(W_{m+k} - \sum_{i=1}^{n_0} |x_i''| \Big) \Big\} = 0 \ .$$

Therefore, we can choose an integer k such that

$$a := \frac{1}{k} \Big(W_{m+k} - \sum_{i=1}^{n_0} |x_i''| \Big) < \min\{|x_i''| : i \in \mathfrak{I}\}.$$

We remark that $x'' \in B_{X''}$ implies $\sum_{i=1}^{n_0} |x''_i| = \sum_{i\in\mathfrak{I}} |x''_i| \leq W_m < W_{m+k}$ for every $k \in \mathbb{N}$ and so $a > 0$. Let

$$y := P_{n_0} x'' + a \sum_{i=n_0+1}^{n_0+k} e_i \ .$$

We claim that $y \in B_X$ and $\sup_{|J|=m+k} \sum_{j\in J} |y_j| = W_{m+k}$ i.e, $\phi_{m+k}(y) = 1$. To see this, we first remark that

$$\|y\| = \sup_{1\leq p\leq m+k} \phi_p(y)$$

since y has only $m+k$ non-zero coordinates.

As $a < \min\{|x''_i| : i \in \mathfrak{I}\}$, we have

$$y^*_m \in \{|x''_i| : i \in \mathfrak{I}\}$$

and so

$$y^*_m > a = |y_{n_0+1}| = ... = |y_{n_0+k}| \ .$$

Hence, for all $1 \leq p \leq m$ we have

$$\max_{|J|=p} \sum_{j\in J} |y_j| \leq \max_{|J|=p} \sum_{j\in J} |x''_j| \leq W_p \ .$$

For $p = m+k$ we have

$$\sup_{|J|=m+k} \sum_{j\in J} |y_j| = \sum_{i=1}^{n_0} |x''_i| + ka = \sum_{i=1}^{n_0} |x''_i| + W_{m+k} - \sum_{i=1}^{n_0} |x''_i| = W_{m+k}.$$

Finally, let $p = m+r$ with $1 \leq r < k$. Clearly

$$k\, W_{m+r} - r\, W_{m+k} = (k-r)\, W_m + (k-r) \sum_{i=m+1}^{m+r} w_i - r \sum_{i=m+r+1}^{m+k} w_i.$$

As w is a decreasing sequence of positive real numbers we have

$$\min\{w_i : m+1 \leq i \leq m+r\} = w_{m+r} \geq w_{m+r+1} = \max\{w_i : m+r+1 \leq i \leq m+k\},$$

and so

$$(k-r) \sum_{i=m+1}^{m+r} w_i - r \sum_{i=m+r+1}^{m+k} w_i \geq (k-r) r w_{m+r} - r(k-r) w_{m+r+1} \geq 0 \ .$$

Consequently,

$$k\, W_{m+r} - r\, W_{m+k} \geq (k-r)\, W_m$$

and so

$$\begin{aligned} W_{m+r} &\geq \frac{1}{k}\Big((k-r)W_m + rW_{m+k}\Big) \\ &\geq \frac{1}{k}\Big((k-r)\sum_{i=1}^{n_0}|x_i''| + rW_{m+k}\Big) \\ &= \sum_{i=1}^{n_0}|x_i''| + r\frac{1}{k}\Big(W_{m+k} - \sum_{i=1}^{n_0}|x_i''|\Big) \\ &= \sum_{i=1}^{n_0}|x_i''| + ra \\ &= \max_{|J|=m+r}\sum_{j\in J}|y_j|. \end{aligned}$$

This completes the proof that $y \in B_X$ and $\phi_{m+k}(y) = 1$.

Summarizing, we have just proved that there exist $k_1 \in \mathbb{N}$ and a positive real number a_1 such that

$$y^{(1)} = P_{n_0}x'' + a_1 \sum_{i=n_0+1}^{n_0+k_1} e_i$$

satisfies $y^{(1)} \in B_X$ and $\phi_{m+k}(y^{(1)}) = 1$. We recall that n_0 is any nonnegative integer satisfying $P_{n_0}(x^{''}) \neq 0$. Continuing the process, we obtain an increasing sequence $(k_j) \subset \mathbb{N}$ and a sequence $(a_j) \subset c_0$ such that $a_j > 0$ for all $j \in \mathbb{N}$ so that the element

$$y^{(j)} = P_{n_0}x'' + a_1 \sum_{i=n+1}^{n+k_1} e_i + a_2 \sum_{i=n+k_1+1}^{n+k_2} e_i + ... + a_j \sum_{i=n+k_{j-1}+1}^{n+k_j} e_i$$

satisfies

$$y^{(j)} \in B_X \text{ and } \phi_{m+k_i}(y^{(j)}) = 1 \quad \text{for all } 1 \leq i \leq j.$$

It is easy to check that the weak*-limit$z^{''}$ of $(y^{(j)})$ is a complex extreme point of $B_{X''}$ in view of Theorem 2.8 and that $P_{n_0}(z^{''}) = P_{n_0}(x'')$. Now, we can repeat this construction in order to get a sequence $(z_n^{''})_{n=1}^{\infty} \subset \mathrm{Ext}_{\mathbb{C}}(B_{X''})$ such that $P_n(z_n'') = P_n(x'')$ for every n large enough. Clearly $(z_n^{''}) \xrightarrow{w^*} x^{''}$ and this completes the proof of $B_{X''} = \overline{\mathrm{Ext}_{\mathbb{C}}(B_{X''})}^{w^*}$. □

Corollary 2.10. *A subset Γ of $B_{d_*(w,1)}$ is a boundary for $\mathcal{A}_{wu}(B_{d_*(w,1)})$ if and only if $B_{d'(w,1)} = \overline{\Gamma}^{w^*}$. As a consequence, if $w \notin \ell_p$ for every $1 \leq p < \infty$, then Γ is a boundary for $\mathcal{A}_u(B_{d'(w,1)})$ if and only if $B_{d'(w,1)} = \overline{\Gamma}^{w^*}$.*

Proof. Since $d(w,1)$ is a separable Banach space, the result follows by using Proposition 2.3 and Proposition 2.9. The second statement follows immediately from Proposition 2.6. □

One way of summarizing a number of these results is to say that Γ is a boundary for $\mathcal{A}_{wu}(B_X)$ if and only if Γ is w^*-dense in the set of complex extreme points of $B_{X''}$. As we have already noted, the set of complex extreme points of B_{ℓ_∞} is itself w^*-closed, whereas Proposition 2.9 shows that the complex extreme points can also be w^*-dense in $B_{X''}$ in some cases.

3. On the Existence of the Minimal Closed Boundary for $\mathcal{A}_\infty(B_X)$

There have been a number of contributions to the study of the existence of the minimal closed boundaries for larger algebras of holomorphic functions defined on the unit ball of a complex Banach space (see [6, 21, 27, 28, 13, 1, 14, 2, 4, 3]). For instance, we know that for $X = c_0,\ d_*(w,1),\ K(H)$ and $\mathcal{C}(K)$ (when H is an infinite dimensional Hilbert space and K is an infinite compact space), there is no minimal closed boundary for $\mathcal{A}_u(B_X)$. In this section we will give a more general result that includes almost all the previous cases. We will concentrate on the Banach algebra $\mathcal{A}_\infty(B_X)$ of all complex valued functions defined on B_X that are continuous and bounded on B_X and holomorphic on the interior $\mathring{B}_X$ of B_X. Clearly $\mathcal{A}_{wu}(B_X) \subset \mathcal{A}_u(B_X) \subset \mathcal{A}_\infty(B_X)$.

In order to do this, it will be convenient to introduce the following property. We say that a Banach space has *property* $(*)$ provided it satisfies the following condition:

There is $0 < r < 1$ such that for every $x_1, x_2 \in B_X$ and $\varepsilon > 0$, there are $y_1, y_2 \in B_X$ and $u \in X$ satisfying $\|y_i - x_i\| < \varepsilon,\ i = 1,2,$ $\quad$ dist $(u, \mathrm{Lin}\ (y_1,y_2)) \geq r$, and $\|y_i + \lambda u\| \leq 1,\ i = 1,2,$ $\quad$ for all $\lambda \in \mathbb{T}$, where $\mathbb{T}$ denotes the set $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

It is easily checked that the spaces $c_0, d_*(w,1)$ and $K(H)$, for an infinite-dimensional complex Hilbert space H satisfy $(*)$, and so some of the results appearing in [2] can be deduced from the following theorem.

Theorem 3.1. *If X is a complex Banach space that satisfies the above property $(*)$, then every closed boundary for $\mathcal{A}_\infty(B_X)$ contains a proper closed subset which is a boundary for $\mathcal{A}_\infty(B_X)$.*

Proof. Given a positive number ε, we first choose $0 < \delta < \min\{1 - r, \frac{r}{4}\}$ and then let $R > 0$ satisfy $\frac{\delta + 1 - r}{R} < \varepsilon$, where $r \in (0,1)$ is the real number in the definition of $(*)$. Assume that Γ is a boundary for $\mathcal{A}_\infty(B_X)$ and let $x_1 \in \Gamma$. Given $h \in \mathcal{A}_\infty(B_X)$, there is $x_2 \in B_X$ such that $|h(x_2)| > \|h\| - \frac{\delta}{R}$. By using the continuity of h and the assumption on X, there are elements $y_i \in B_X$ $(i = 1,2)$ and $u \in X$ such that $\|y_i - x_i\| < \delta,\quad |h(y_2)| > \|h\| - \frac{\delta}{R},\quad \|y_i + \lambda u\| \leq 1,$ for all $\lambda \in \mathbb{T},$ and dist $(u, \mathrm{Lin}\ (y_1, y_2)) \geq r$.

We choose $u_0' \in S_{X'}$ such that $u_0'(y_i) = 0$ for $i = 1,2$ and $|u_0'(u)| \geq r$. Let λ_0 be a complex number with $|\lambda_0| = 1$, chosen so that

$$|h(y_2 + \lambda_0 u)| \geq |h(y_2)| \ . \tag{4}$$

Now we choose $\mu_0 \in \mathbb{C}$ satisfying $|\mu_0| = 1$ and

$$|Rh(y_2 + \lambda_0 u) + \mu_0 u_0'(\lambda_0 u)| = |Rh(y_2 + \lambda_0 u)| + |u_0'(u)| \ .$$

We define the function g by

$$g(x) := Rh(x) + \mu_0 u_0'(x) \qquad (x \in B_X) \ .$$

Since $h \in \mathcal{A}_\infty(B_X)$, g also belongs to $\mathcal{A}_\infty(B_X)$ and we have

$$\begin{aligned} \|g\| &\geq |g(y_2 + \lambda_0 u)| \\ &= |Rh(y_2 + \lambda_0 u)| + |u_0'(u)| \quad \text{(by (4))} \\ &\geq |Rh(y_2)| + |u_0'(u)| \quad \text{(by the choice of } u_0' \text{ and } y_2) \\ &> R\|h\| - \delta + r \ . \end{aligned}$$

Since Γ is a boundary for $\mathcal{A}_\infty(B_X)$, there is $v \in \Gamma$ such that $|g(v)| > R\|h\| - \delta + r$. Hence we have

$$R\|h\| + |u_0'(v)| \geq R|h(v)| + |u_0'(v)| \geq |g(v)| > R\|h\| - \delta + r \ , \tag{5}$$

and so,

$$|u_0'(v)| > r - \delta \ .$$

By the choice of y_1 and δ we have

$$\begin{aligned} \|v - x_1\| &\geq \|v - y_1\| - \|y_1 - x_1\| \\ &> |u_0'(v - y_1)| - \delta \\ &= |u_0'(v)| - \delta \\ &\geq r - 2\delta > \frac{r}{2} \ . \end{aligned}$$

In view of (5) we also obtain

$$R|h(v)| + 1 \geq R|h(v)| + |u_0'(v)| > R\|h\| - \delta + r$$

and so

$$|h(v)| > \|h\| + \frac{-\delta + r - 1}{R} > \|h\| - \varepsilon \ .$$

We have proved that $\Gamma \backslash (x_1 + \frac{r}{2} B_X)$ is also a boundary for $\mathcal{A}_\infty(B_X)$. $\square$

The following is a consequence of the above proof.

Corollary 3.2. *Let X be a complex Banach space that satisfies the above property $(*)$. There is a positive number s satisfying that if $\Gamma \subset B_X$ is a boundary for $\mathcal{A}_\infty(B_X)$ and $x_1 \in \Gamma$, then $\Gamma \backslash (x_1 + s \overset{\circ}{B}_X)$ is also a boundary. Therefore, there is no minimal closed boundary for $\mathcal{A}_\infty(B_X)$. The same result also holds for every subalgebra $\mathcal{A} \subset \mathcal{A}_\infty(B_X)$ that contains the 1-degree polynomials on X.*

We now provide a wider class where the same result can be applied.

Definition 3.3. ([23, Definition II.4.1]) A Banach space is said to have the *intersection property* if for every $\varepsilon > 0$, there is a finite set $G \subset S_X$ and $\delta > 0$ such that whenever $x \in X$ is such that $\|z \pm x\| \leq 1 + \delta$ for all $z \in G$, then $\|x\| \leq \varepsilon$.

One large class having the intersection property are those Banach spaces having the Radon-Nikodým property (see [23, Proposition II.4.2]). However, as we will see, spaces that lack the intersection property are also of interest.

The following implication is easy to verify and will be useful in the proof of the next lemma.

$$\text{If } \|x \pm z\| \leq 1, \quad \text{then} \quad \|x \pm tz\| \leq 1 \text{ for all } t \in [-1,1] \tag{6}$$

Lemma 3.4. *A Banach space X lacks the intersection property if and only if for some $r \in (0,1)$ the following property holds: For every finite set $F \subset \overset{\circ}{B}_X$ there is some $u \in X$, $\|u\| > r$, such that for all $z \in F$,*

$$\|z \pm u\| \leq 1.$$

Proof. If X fails the intersection property, then for some $0 < \varepsilon_0 < \frac{1}{2}$ we have that for any finite subset G of S_X and $\delta > 0$ there exists $x \in X$ satisfying $\|x\| > \varepsilon_0$ and $\|z \pm x\| \leq 1+\delta$, for all $z \in G$. Take $F \subset \overset{\circ}{B}_X$, finite and let $G := \left\{\frac{z}{\|z\|} : z \in F, z \neq 0\right\} \subset S_X$. Choose $s := \max\{\|z\| : z \in F\} < 1$ and apply the hypothesis to G and $0 < \delta < \min\{\frac{1}{s} - 1, 1\}$ to get an element $x \in X$, $\|x\| > \varepsilon_0$, such that for all $0 \neq z \in F$,

$$\left\|\frac{z}{\|z\|} \pm x\right\| \leq 1 + \delta.$$

Now it suffices to take $u = \frac{x}{1+\delta}$ and $r := \frac{\varepsilon_0}{2} < \frac{\varepsilon_0}{1+\delta}$ and use (6) to show that for all $0 \neq z \in F$,

$$\|z \pm u\| \leq 1.$$

We can clearly take $\|u\| \leq 1$, so, if $0 \in F$, the above inequality is also satisfied for $z = 0$.

Conversely, suppose that for some $r > 0$ we have that given any finite subset F of $\overset{\circ}{B}_X$, there exists $u \in X$, $\|u\| > r$, that satisfies

$$\|z \pm u\| \leq 1,$$

for all $z \in F$.

Given a finite subset $G \subset S_X$ and $\delta > 0$, take $F := \frac{G}{1+\delta} \subset \overset{\circ}{B}_X$. By hypothesis, there is $u \in X$, $\|u\| > r$, such that for all $x \in G$,

$$\left\|\frac{x}{1+\delta} \pm u\right\| \leq 1 \ .$$

So, for all $x \in G$,

$$\|x \pm u(1+\delta)\| \leq 1 + \delta,$$

and $\|u(1+\delta)\| \geq r(1+\delta) > r$. Hence X does not have the intersection property. $\square$

We now give a broad class of Banach spaces to which Theorem 3.1 can be applied. In order to do this, we will relate the intersection property and property $(*)$.

Proposition 3.5. *If X is a Banach space not satisfying the intersection property, then X satisfies property $(*)$.*

Proof. Since X does not have the intersection property, there is $r > 0$ satisfying the condition stated in the previous lemma, so we know that $r < 1$. Now, if $x_1, x_2 \in B_X$ and $\varepsilon > 0$, we choose $\eta > 0$ such that $2\eta < \min\{\varepsilon, \frac{r}{4}\}$ and let $z_i := \frac{1}{1+\eta} x_i$ $(i = 1, 2)$. Hence there is $u \in X$ such that

$$r < \|u\| \quad \text{and} \quad \|\mathrm{z} \pm \mathrm{u}\| \leq 1 \tag{7}$$

for all $z \in F$, where $F \subset \frac{1}{1+\eta} B_M$ is a finite set such that

$$\{z_1, z_2\} \subset F \quad \text{and} \quad \frac{1}{1+\eta} B_M \subset F + \eta B_X \ , \tag{8}$$

and $M := \text{Lin}\ \{x_1, x_2\}$.

For each $\lambda \in \mathbb{T}$, $\lambda z_i \in \frac{1}{1+\eta} B_M$ $(i = 1, 2)$ and so for some $z \in F$, $\|\lambda z_i - z\| \leq \eta$. Then

$$\|\lambda z_i + u\| \leq \|\lambda z_i - z\| + \|z + u\| \leq 1 + \eta \ . \tag{9}$$

If we take $y_i := \frac{z_i}{1+\eta} = \frac{x_i}{(1+\eta)^2}$ $(i = 1, 2)$ and $v := \frac{u}{1+\eta}$, we also obtain that

$$\|y_i - x_i\| \leq \|y_i - z_i\| + \|z_i - x_i\| \leq 2\eta < \varepsilon$$

and we deduce from inequality (9) that

$$\|\lambda y_i + v\| \leq 1,$$

for all $\lambda \in \mathbb{T}$.

Let us write $d := \text{dist}\ (u, M)$. There is $m_0 \in B_M$ such that $\|u - m_0\| = d$. If $m_0 = 0$, then $d = \|u\| > r$. Otherwise, by using (8) and (7) we have $\left\| \frac{m_0}{\|m_0\|} \pm u \right\| \leq 1 + 2\eta$. If $m_0' \in S_{X'}$ satisfies $m_0'(m_0) = \|m_0\|$, then $|\text{Re}\ m_0'(u)| \leq 2\eta$.

If $m_0' \in S_{X'}$ satisfies $m_0'(m_0) = \|m_0\|$, then $|\text{Re}\ m_0'(u)| \leq 2\eta$.

From the inequality

$$\big| \|m_0\| - m_0'(u) \big| = |m_0'(m_0 - u)\| \leq \|m_0 - u\| = d$$

we obtain

$$\|m_0\| - d \leq |\text{Re}\ \ m_0'(u)| \leq 2\eta,$$

that is,

$$\|m_0\| - 2\eta \leq d \ .$$

If $\|m_0\| < \dfrac{r}{2}$, then

$$d = \|u - m_0\| \geq \|u\| - \|m_0\| > r - \frac{r}{2} = \frac{r}{2} \ .$$

Otherwise

$$d \geq \|m_0\| - 2\eta \geq \frac{r}{2} - 2\eta > \frac{r}{4} \ .$$

In any case we obtain $d > \frac{r}{4}$ and so dist $(v, M) > \frac{r}{4(1+\eta)} > \frac{r}{5}$.

□

We recall that a Banach space X is an *M-ideal in its bidual* if there is a decomposition $X''' = X' \oplus_1 X^{\perp}$, that is,

$$\|x' + x'''\| = \|x'\| + \|x'''\|,$$

for all $x' \in X', x''' \in X''' \cap X^{\perp}$. X is said to be a *proper M-ideal* in X'' if it is an M-ideal in X'' and it is not reflexive.

Corollary 3.6. *Assume that X is a complex Banach space which is a proper M-ideal in X'' and let $\mathcal{A}$ be an arbitrary subalgebra of $\mathcal{A}_\infty(B_X)$ containing the 1-degree polynomials on X. Then every closed boundary for $\mathcal{A}$ contains a proper closed subset which is a boundary for $\mathcal{A}$. In particular, there is no minimal closed boundary for $\mathcal{A}$.*

Proof. It is known that a proper M-ideal in its bidual does not have the intersection property [23, Theorem II.4.4], so in view of Proposition 3.5, it satisfies property $(*)$. By Corollary 3.2, there is no minimal closed boundary for $\mathcal{A}$ for every subalgebra $\mathcal{A}$ of $\mathcal{A}_\infty(B_X)$ containing the 1-degree polynomials on X. $\square$

Since $d_*(w,1)$ is a proper M-ideal in its bidual (see [33, Proposition 2.2] or [23, Examples III.1.4c]), we deduce that there is no minimal closed boundary for $\mathcal{A}_{wu}(B_{d_*(w,1)})$.

The next result is due to Whitfield and Zizler in the real case and it provides a class of spaces to which the last theorem can also be applied. The same argument used in the proof of [34, Theorem 1.c] also works for the complex case.

Proposition 3.7. *If $(X, \| \ \|)$ is a Banach space that contains a complemented copy of c_0, then there exists an equivalent norm $||| \ |||$ in X such that $(X, ||| \ |||)$ has the following property. If K is a compact subset of the closed unit ball B_1 of $(X, ||| \ |||)$ and $0 < s < 1$ then there exists $u \in X$ such that $|||u||| > s$ and $K + u \subset B_1$. Hence $(X, ||| \ |||)$ satisfies property $(*)$.*

As a consequence of Proposition 3.7 and Corollary 3.2 we obtain the following result:

Corollary 3.8. *If $(X, \| \ \|)$ contains a complemented copy of c_0, there exists an equivalent norm $||| \ |||$ in X so that if B_1 is the closed unit ball of $(X, ||| \ |||)$ then every closed boundary for $\mathcal{A}_\infty(B_1)$ contains a proper closed subset which is a boundary for $\mathcal{A}_\infty(B_1)$. In fact, there is a positive number r such that if $\Gamma \subset B_1$ is a boundary for $\mathcal{A}_\infty(B_1)$ and $z \in \Gamma$, then $\Gamma \backslash (z + r\overset{\circ}{B}_1)$ is also a boundary. Indeed, in this case, r can be taken arbitrarily close to 1.*

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