CONSTRUCTION OF WEAKLY DENSE, NORM DIVERGENT SEQUENCES

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Abstract. Let X be a separable Banach space. We provide an explicit construction of a sequence in X that tends to ∞ in norm but which is weakly dense.

Our interest in the result stated in the Abstract was motivated by two theorems. First, in their work on hypercyclic operators, K. Chan and R. Sanders proved the following:

Theorem 1. (Chan and Sanders [3]) *For any p, $2 \leq p < \infty$, there is a bounded linear operator $T : \ell_p(\mathbb{Z}) \to \ell_p(\mathbb{Z})$ that is weakly hypercyclic but is not hypercyclic.*

That is, there is a vector $x_p \in \ell_p(\mathbb{Z})$ such that $\{x_p, T(x_p), \cdots, T^n(x_p), \cdots\}$ is a weakly dense set and such that for no vector $x \in \ell_p(\mathbb{Z})$ is it true that $\{x, T(x), \cdots, T^n(x), \cdots\}$ is norm dense. In fact, the proof in [3] shows the existence of a vector x_p such that $\|T^n(x_p)\| \to \infty$ while $\{T^n(x_p) \mid n \in \mathbb{N}\}$ is dense in $\ell_p(\mathbb{Z})$ with the weak topology. Moreover, Chan and Sanders remark that, in fact, one can construct a weakly dense sequence $(x_n) \subset \ell_2(\mathbb{Z})$ such that $\|x_n\| \to \infty$ ([3, page 49]).

Second, V. Kadets has proved the following result (see also [6]):

Theorem 2. (Kadets [7]) *For any Banach space X, for any sequence (c_n) of positive real numbers such that $\sum_{n=1}^{\infty} c_n^{-2} = \infty$, there is a sequence $(x_n) \subset X$, $\|x_n\| = c_n$ for all n, such that 0 is in the weak closure of the sequence $\{x_n \mid n \in \mathbb{N}\}$.*

The proof of this theorem uses Dvoretzky's theorem [4] and consequently is non-constructive. In fact, S. Shkarin [8] recently rediscovered the Kadets result; his proof also makes use of Dvoretzky's theorem.

In this note, we show that Kadets' result yields, as a simple consequence, that for any separable Banach space X, there is a sequence $(x_n) \subset X$ such that $\|x_n\| \to \infty$ while $\{x_n \mid n \in \mathbb{N}\}$ is weakly dense in X.

Moreover, we give a version of Theorem 2 whose proof provides a rather simple, explicit sequence of vectors and which uses the pigeon hole principle instead of Dvoretzky's theorem.

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To begin, we show that the conclusion of Theorem 2 can be strengthened.

Theorem 3. *Let X be a separable Banach space. Suppose that there is a sequence $(x_n) \subset X$ with the following two properties:*

(1) $\|x_n\| \to \infty$ *as* $n \to \infty$,
(2) 0 *is in the weak closure of* $\{x_n \mid n \in \mathbb{N}\}$.

Then there is a sequence $(y_n) \in X$ such that the following hold:

(1) $\|y_n\| \to \infty$ *as* $n \to \infty$,

(2) $\{y_n \mid n \in \mathbb{N}\}$ *is weakly dense in* X.

Proof. Consider any sequence (z_r) that is norm dense in X. Let (x_n) be a sequence in $X \setminus \{0\}$ such that $(\|x_n\|)$ diverges to ∞ and 0 belongs to the weak closure of (x_n). Let $x_n^* \in X^*$ of norm 1 such that $x_n^*(x_n) = \|x_n\|$. Let $\gamma_{n,r}$ defined by

$$\gamma_{n,r} = \begin{cases} \frac{x_n^*(z_r)}{|x_n^*(x_r)|} & \text{if } x_n^*(z_r) \neq 0 \\ 1 & \text{if } x_n^*(z_r) = 0. \end{cases}$$

Then

$$\|z_r + \gamma_{n,r}x_n\| \geq |x_n^*(z_r + \gamma_{n,r}x_n)| = |x_n^*(x_r)| + \|x_n\| \geq \|x_n\|,$$

for all $r \leq n \in \mathbb{N}$. We apply the square ordering to

$$\begin{array}{cccccc}
z_1 + \gamma_{1,1}x_1 & \rightarrow & z_1 + \gamma_{2,1}x_2 & & z_1 + \gamma_{3,1}x_3 & \rightarrow \\
& \swarrow & & \nearrow & & \swarrow \\
z_2 + \gamma_{2,2}x_2 & & z_2 + \gamma_{3,2}x_3 & & z_2 + \gamma_{4,2}x_4 & \\
\downarrow & \nearrow & & \swarrow & & \\
z_3 + \gamma_{1,3}x_3 & & z_3 + \gamma_{4,3}x_4 & & z_3 + \gamma_{5,3}x_5 & \cdots \\
\cdots & \cdots & \cdots & \cdots & &
\end{array}$$

obtaining a sequence (y_s). Since $(\|x_n\|)$ diverges to ∞, then $(\|y_s\|)$ diverges to ∞ too. Moreover,

$$z_r \in \overline{\{y_s : s \in \mathbb{N}\}}^{w(X,X^*)},$$

for all $r \in \mathbb{N}$ and hence $\overline{\{y_s : s \in \mathbb{N}\}}^{w(X,X^*)} = X$. $\square$

The rest of this note is devoted to providing an explicit construction of a sequence (x_n) satisfying the conclusion of Theorem 2, for any separable Banach space X. We begin with $X = \ell_1$, considered as a real Banach space. Our argument uses the pigeon hole principle.

Proposition 4. *Consider the sequence (x_n) in ℓ_1 obtained by any ordering of the set $\cup_{k=1}^{\infty}\{\sqrt{k}(e_{m_1} - e_{m_2}) \ : \ 1 \leq m_1 < m_2 \leq (2k)^{k+1}\}$. We have that $(\|x_n\|)$ diverges to ∞ and 0 belongs to the weak closure of (x_n).*

Proof. Consider a natural number k and $\varphi_1, \dots, \varphi_k$ in the unit sphere of ℓ_∞. We denote $K_0 = \{1, \dots, (2k)^{k+1}\}$. If

$$\varphi_j = (\alpha_m^j)_{m=1}^\infty$$

we put

$$J_p^1 = \{m \le (2k)^{k+1} : \alpha_m^1 \in [\frac{p}{k}, \frac{p+1}{k}]\}$$

for $-k \le p \le k-1$, then

$$\{1, \dots, (2k)^{k+1}\} = \bigcup_{p=-k}^{k-1} J_p^1.$$

Thus there exists a p_1, $-k \le p_1 \le k-1$, such that

$$\mathrm{Card}(J_{p_1}^1) \ge (2k)^k.$$

Choose a subset $K_1 \subset J_{p_1}^1$ such that $\mathrm{Card}(K_1) = (2k)^k$. Now we define

$$J_p^2 = \{m \in K_1 : \alpha_m^2 \in [\frac{p}{k}, \frac{p+1}{k}]\}$$

for $-k \le p \le k-1$, then

$$K_1 = \bigcup_{p=-k}^{k-1} J_p^2$$

and $\mathrm{Card}(K_1) = (2k)^k$. Thus there exists a p_2, $-k \le p_2 \le k-1$, such that $\mathrm{Card}(J_{p_2}^2) \ge (2k)^{k-1}$, and as before we let $K_2 \subset J_{p_2}^2$ such that $\mathrm{Card}(K_2) = (2k)^{k-1}$. We continue by induction. For $l < k$, we assume that $(K_j)_{j=1}^l$ and $(p_j)_{j=1}^l$ have been found so that $K_l \subset K_{l-1} \subset \ldots \subset K_2 \subset K_1$, $-k \le p_j \le k-1$,

$$K_j \subset \{m \in K_{j-1} : \alpha_m^j \in [\frac{p_j}{k}, \frac{p_j+1}{k}]\}$$

and $\mathrm{Card}(K_j) = (2k)^{k-j+1}$, for $j = 1, \dots, l$. Since

$$k - l + 1 \ge 2$$

if we define again

$$J_p^{l+1} = \{m \in K_l : \alpha_m^{l+1} \in [\frac{p}{k}, \frac{p+1}{k}]\}$$

for $-k \le p \le k-1$, then $K_l = \bigcup_{p=-k}^{k-1} J_p^{l+1}$ and $\mathrm{Card}(K_l) = (2k)^{k-l+1} \ge (2k)^2$. Thus there exists a p_{l+1}, $-k \le p_{l+1} \le k-1$, such that $\mathrm{Card}(J_{p_{l+1}}^{l+1}) \ge (2k)^{k-l}$. Again we consider $K_{l+1} \subset J_{p_{l+1}}^{l+1}$ such that $\mathrm{Card}(K_{l+1}) = (2k)^{k-l}$.

By taking $l = k-1$, we obtain finally the existence of a K_k with

$$K_k \subset K_{k-1} \subset \ldots \subset K_2 \subset K_1 \subset \{1, \dots, (2k)^{k+1}\},$$

$$\mathrm{Card}(K_k) = 2k,$$

and

$$|\alpha_m^j - \alpha_r^j| \leq \frac{1}{k}$$

for all $j = 1, \dots, k$ and all $m, r \in K_k$. Since $\mathrm{Card}(K_k) = 2k \geq 2$, we can take $m_1 < m_2$, such that

$$\{m_1, m_2\} \subset K_k.$$

Consider

$$x = \sqrt{k}(e_{m_1} - e_{m_2}).$$

We have that

$$|\varphi_j(x)| = |\sqrt{k}(\alpha_{m_1}^j - \alpha_{m_2}^j)| \leq \frac{\sqrt{k}}{k} = \frac{1}{\sqrt{k}}, \tag{0.1}$$

for all $j = 1, \dots, k$ and

$$\|x\|_1 = 2\sqrt{k}.$$

Let

$$I_k = \{\sqrt{k}(e_{m_1} - e_{m_2}) \ : \ 1 \leq m_1 < m_2 \leq (2k)^{k+1}\}$$

for $k = 1, 2, \dots$. We take

$$(x_n)_{n=1}^{\infty} = \bigcup_{k=1}^{\infty} I_k$$

where (e.g.) we first order the elements of I_1, then of I_2 and so on. Clearly $(\|x_n\|)$ diverges to ∞. We claim that 0 belongs to the weak closure of (x_n). Indeed, given $\phi_1, \dots, \phi_h$ in $\ell_\infty \setminus \{0\}$ and $\varepsilon > 0$, we consider $k \geq h$ such that

$$\frac{\sup\{\|\phi_1\|, \dots, \|\phi_h\|\}}{\sqrt{k}} < \varepsilon,$$

and we define

$$\varphi_j = \begin{cases} \frac{\phi_j}{\|\phi_j\|} & \text{if } 1 \leq j \leq h \\ \frac{\phi_h}{\|\phi_h\|} & \text{if } h \leq j \leq k. \end{cases} \tag{0.2}$$

By (0.1) we can find an $x_n = \sqrt{k}(e_{m_1} - e_{m_2})$ for a certain pair $\{m_1, m_2\}$, $1 \leq m_1 < m_2 \leq (2k)^{k+1}$ such that

$$|\frac{\phi_j}{\|\phi_j\|}(x_n)| \leq \frac{1}{\sqrt{k}}.$$

for all $j = 1, \dots, h$. Hence

$$|\phi_j(x_n)| \leq \frac{\|\phi_j\|}{\sqrt{k}} < \varepsilon,$$

for all $j = 1, \dots, h$, and we have obtained

$$0 \in \overline{\{x_n : n \in \mathbb{N}\}}^{w(\ell_1, \ell_\infty)}.$$

$\square$

Note that the sequence (x_n) that was constructed in the above proof is particularly simple; namely each x_n is of the form $C(e_i - e_j)$. We now show how the previous proposition yields the general result. Note that given a dense sequence in a separable Banach space, the sequence (z_n) in the following Corollary can be explicitly described.

Corollary 5. *Given a separable Banach space X, there is a sequence $(z_n) \subset X$ such that $\|z_n\| \to \infty$ and $X = \overline{\{z_n \mid n \in \mathbb{N}\}}^{w(X,X^*)}$.*

Proof. By Theorem 3, it is enough to prove that there exists a sequence (z_n) in $X \setminus \{0\}$ such that $(\|z_n\|)$ diverges to ∞ and 0 belongs to the weak closure of (z_n). Let B_X be the open unit ball of X. Since X is infinite dimensional, the Riesz lemma allows us to construct a sequence (y_n) in B_X such that $\|y_n - y_m\| \geq \frac{1}{2}$ for all $n \neq m$.

Now we define $T : \ell_1 \longrightarrow X$ by

$$T(\alpha_n) = \sum_{n=1}^{\infty} \alpha_n y_n.$$

Since $T(B_{\ell_1}) \subset B_X$, T is continuous. Let $(x_n) \subset \ell_1$ be the sequence obtained in Proposition 4. We are going to check that $z_n = T(x_n)$ defines the sequence that we are looking for. If $A = \{x_n \ : \ n \in \mathbb{N}\}$, we know that $0 \in \overline{A}^{w(\ell_1,\ell_\infty)}$. Since T is weak-weak continuous, we get

$$0 = T(0) \in T(\overline{A}^{w(\ell_1,\ell_\infty)}) \subset \overline{T(A)}^{w(X,X^*)}.$$

Moreover, we know that for each n there exist unique $k(n), m_1(n), m_2(n) \in \mathbb{N}$ with $m_1(n) \neq m_2(n)$ such that $x_n = \sqrt{k(n)}(e_{m_1(n)} - e_{m_2(n)})$ and $k(n) \to \infty$ whenever n grows to ∞. Hence $\|x_n\|_1 = \sqrt{2k(n)}$ diverges to ∞. But

$$\begin{aligned} \|T(x_n)\| &= \sqrt{k(n)}\|T(e_{m_1(n)}) - T(e_{m_2(n)})\| \\ &= \sqrt{k(n)}\|y_{m_1(n)} - y_{m_2(n)}\| \geq \frac{1}{2}\sqrt{k(n)}\ , \end{aligned}$$

for all $n \in \mathbb{N}$ and we have obtained that $\|(z_n)\|$ diverges to ∞ too. □

Comments.

(1) Proposition 4 can be adapted to the complex ℓ_1 case. As a consequence, Corollary 5 holds for $\mathbb{R}$ and $\mathbb{C}$.
(2) The above argument shows that for every infinite dimensional Banach space (separable or not) one can construct a sequence (x_n) in $X \setminus \{0\}$ such that $(\|x_n\|)$ diverges to ∞ and such that 0 belongs to the weak closure of (x_n).

(3) Corollary 5 is much weaker than Theorem 1. Indeed, the result of [3] produces an operator $T : \ell_p(\mathbb{Z}) \to \ell_p(\mathbb{Z}),\ p \geq 2$ which, in turn, produces the sequence (x_n). Shkarin notes there is no bilateral shift on $\ell_p(\mathbb{Z})$ for $1 \leq p < 2$ that is weakly, but not norm, hypercyclic [8]. Although this does not rule out the possibility of finding a bounded operator T on such spaces and a weakly dense sequence (x_n) such that $T(x_n) = x_{n+1}$ for all n, it at least gives an indication that, if they exist, such operators are difficult to come by. (See also related work of S. Grivaux [5].)
(4) Using a result of Ball [1, Theorem 7], Shkarin ([8, Proposition 5.4]) shows that if the norms of the sequence (x_n) tend "too rapidly" to infinity (e.g. if $\sum_n \|x_n\|^{-1} < \infty$), then $0 \notin \overline{\{x_n \mid n \in \mathbb{N}\}}^{w(X,X^*)}$.

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