APOS: A Constructivist Theory of Learning
in Undergraduate Mathematics Education Research

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The work reported in this paper is based on the principle that research in mathematics education is strengthened in several ways when based on a theoretical perspective. Development of a theory or model in mathematics education should be, in our view, part of an attempt to understand how mathematics can be learned and what an educational program can do to help in this learning. We do not think that a theory of learning is a statement of truth and although it may or may not be an approximation to what is really happening when an individual tries to learn one or another concept in mathematics, this is not our focus. Rather we concentrate on how a theory of learning mathematics can help us understand the learning process by providing explanations of phenomena that we can observe in students who are trying to construct their understandings of mathematical concepts and by suggesting directions for pedagogy that can help in this learning process.

Models and theories in mathematics education can

- support prediction,

- have explanatory power,

- be applicable to a broad range of phenomena,

- help organize one's thinking about complex, interrelated phenomena,

- serve as a tool for analyzing data, and

- provide a language for communication of ideas about learning that go beyond superficial descriptions.

We would like to offer these six features, the first three of which are given by Alan Schoenfeld in "Toward a theory of teaching-in-context," *Issues in Education*, both as ways in which a theory can contribute to research and as criteria for evaluating a theory.

In this paper, we describe one such perspective, APOS Theory, in the context of undergraduate mathematics education. We explain the extent to which it has the above characteristics, discuss the role that this theory plays in a research and curriculum development program and how such a program can contribute to the development of the theory, describe briefly how working with this particular theory has provided a vehicle for building a community of researchers in undergraduate mathematics education, and indicate the use of APOS Theory in specific research studies, both by researchers who are developing it as well as others not connected with its development. We provide, in connection with this paper, an annotated bibliography of research reports which involve this theory.

APOS Theory

The theory we present begins with the hypothesis that mathematical knowledge consists in an individual's tendency to deal with perceived mathematical problem situations by constructing mental *actions*, *processes*, and *objects* and organizing them in *schemas* to make sense of the situations and solve the problems. In reference to these mental constructions we call it *APOS Theory*. The ideas arise from our attempts to extend to the level of collegiate mathematics learning the work of J. Piaget on reflective abstraction in children's learning. APOS Theory is discussed in detail in Asiala, et. al. (1996). We will argue that this theoretical perspective possesses, at least to some extent, the characteristics listed above and, moreover, has been very useful in attempting to understand students' learning of a broad range of topics in calculus, abstract algebra, statistics, discrete mathematics, and other areas of undergraduate mathematics. Here is a brief summary of the essential components of the theory.

An *action* is a transformation of objects perceived by the individual as essentially external and as requiring, either explicitly or from memory, step-by-step instructions on how to perform the operation. For example, an individual with an action conception of left coset would be restricted to working with a concrete group such as Z_{20} and he or she could construct subgroups, such as $H=\{0,4,8,12,16\}$ by forming the multiples of 4. Then the individual could write the left coset of 5 as the set $5+H=\{1,5,9,13,17\}$ consisting of the elements of Z_{20} which have remainders of 1 when divided by 4.

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When an action is repeated and the individual reflects upon it, he or she can make an internal mental construction called a *process* which the individual can think of as performing the same kind of action, but no longer with the need of external stimuli. An individual can think of performing a process without actually doing it, and therefore can think about reversing it and composing it with other processes. An individual cannot use the action conception of left coset described above very effectively for groups such as S_4, the group of permutations of four objects and the subgroup H corresponding to the 8 rigid motions of a square, and not at all for groups S_n for large values of n. In such cases, the individual must think of the left coset of a permutation p as the set of all products ph, where h is an element of H. Thinking about forming this set is a process conception of coset.

An *object* is constructed from a process when the individual becomes aware of the process as a totality and realizes that transformations can act on it. For example, an individual understands cosets as objects when he or she can think about the number of cosets of a particular subgroup, can imagine comparing two cosets for equality or for their cardinalities, or can apply a binary operation to the set of all cosets of a subgroup.

Finally, a *schema* for a certain mathematical concept is an individual's collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept. This framework must be coherent in the sense that it gives, explicitly or implicitly, means of determining which phenomena are in the scope of the schema and which are not. Because this theory considers that all mathematical entities can be represented in terms of actions, processes, objects, and schemas, the idea of schema is very similar to the concept image which Tall and Vinner introduce in "Concept image and concept definition in mathematics with particular reference to limits and continuity," *Educational Studies in Mathematics*, **12**, 151-169 (1981). Our requirement of coherence, however, distinguishes the two notions.

The four components, action, process, object, and schema have been presented here in a hierarchical, ordered list. This is a useful way of talking about these constructions and, in some sense, each conception in the list must be constructed before the next step is possible. In reality, however, when an individual is developing her or his understanding of a concept, the constructions are not

actually made in such a linear manner. With an action conception of function, for example, an individual may be limited to thinking about formulas involving letters which can be manipulated or replaced by numbers and with which calculations can be done. We think of this notion as preceding a process conception, in which a function is thought of as an input-output machine. What actually happens, however, is that an individual will begin by being restricted to certain specific kinds of formulas, reflect on calculations and start thinking about a process, go back to an action interpretation, perhaps with more sophisticated formulas, further develop a process conception and so on. In other words, the construction of these various conceptions of a particular mathematical idea is more of a dialectic than a linear sequence.

APOS Theory can be used directly in the analysis of data by a researcher. In very fine grained analyses, the researcher can compare the success or failure of students on a mathematical task with the specific mental constructions they may or may not have made. If there appear two students who agree in their performance up to a very specific mathematical point and then one student can take a further step while the other cannot, the researcher tries to explain the difference by pointing to mental constructions of actions, processes, objects and/or schemas that the former student appears to have made but the other has not. The theory then makes testable predictions that if a particular collection of actions, processes, objects and schemas are constructed in a certain manner by a student, then this individual will likely be successful using certain mathematical concepts and in certain problem situations. Detailed descriptions, referred to as *genetic decompositions,* of schemas in terms of these mental constructions are a way of organizing hypotheses about how learning mathematical concepts can take place. These descriptions also provide a language for talking about such hypotheses.

Development of APOS Theory

APOS Theory arose out of an attempt to understand the mechanism of *reflective abstraction*, introduced by Piaget to describe the development of logical thinking in children, and extend this idea to more advanced mathematical concepts (Dubinsky, 1991a). This work has been carried on by a small group of researchers called a Research in Undergraduate Mathematics Education Community (RUMEC) who have been collaborating on specific research projects using APOS Theory within a

broader research and curriculum development framework. The framework consists of essentially three components: a theoretical analysis of a certain mathematical concept, the development and implementation of instructional treatments (using several non-standard pedagogical strategies such as cooperative learning and constructing mathematical concepts on a computer) based on this theoretical analysis, and the collection and analysis of data to test and refine both the initial theoretical analysis and the instruction. This cycle is repeated as often as necessary to understand the epistemology of the concept and to obtain effective pedagogical strategies for helping students learn it.

The theoretical analysis is based initially on the general APOS theory and the researcher's understanding of the mathematical concept in question. After one or more repetitions of the cycle and revisions, it is also based on the fine-grained analyses described above of data obtained from students who are trying to learn or who have learned the concept. The theoretical analysis proposes, in the form of a genetic decomposition, a set of mental constructions that a student might make in order to understand the mathematical concept being studied. Thus, in the case of the concept of cosets as described above, the analysis proposes that the student should work with very explicit examples to construct an action conception of coset; then he or she can interiorize these actions to form processes in which a (left) coset gH of an element g of a group G is imagined as being formed by the process of iterating through the elements h of H, forming the products gh, and collecting them in a set called gH; and finally, as a result of applying actions and processes to examples of cosets, the student encapsulates the process of coset formation to think of cosets as objects. For a more detailed description of the application of this approach to cosets and related concepts, see Asiala, Dubinsky, et. al. (1997).

Pedagogy is then designed to help the students make these mental constructions and relate them to the mathematical concept of coset. In our work, we have used cooperative learning and implementing mathematical concepts on the computer in a programming language which supports many mathematical constructs in a syntax very similar to standard mathematical notation. Thus students, working in groups, will express simple examples of cosets on the computer as follows.

```
Z20 := {0..19};

op := |(x,y) -> x+y (mod 20)|;
```

```
H := {0,4,8,12,16};

5H := {1,5,9,13,17};
```

To interiorize the actions represented by this computer code, the students will construct more complicated examples of cosets, such as those appearing in groups of symmetries.

```
Sn := {[a,b,c,d] : a,b,c,d in {1,2,3,4} | #{a,b,c,d} = 4};

op := |(p,q) -> [p(q(i)) : i in [1..4]]|;

H := {[1,2,3,4], [2,1,3,4], [3,4,1,2], [4,3,2,1]};

p := [4,3,2,1];

pH := {p .op q : q in H};
```

The last step, to encapsulate this process conception of cosets to think of them as objects, can be very difficult for many students. Computer activities to help them may include forming the set of all cosets of a subgroup, counting them, and picking two cosets to compare their cardinalities and find their intersections. These actions are done with code such as the following.

```
SnModH := {{p .op q : q in H} : p in Sn};

#SnModH;

L := arb(SnModH); K := arb(SnModH); #L = #K;  L inter K;
```

Finally, the students write a computer program that converts the binary operation op from an operation on elements of the group to subsets of the group. This structure allows them to construct a binary operation (coset product) on the set of all cosets of a subgroup and begin to investigate quotient groups.

It is important to note that in this pedagogical approach, almost all of the programs are written by the students. One hypothesis that the research investigates is that, whether completely successful or not, the task of writing appropriate code leads students to make the mental constructions of actions, processes, objects, and schemas proposed by the theory. The computer work is accompanied by classroom discussions that give the students an opportunity to reflect on what they have done in the computer lab and relate them to mathematical concepts and their properties and relationships. Once the concepts are in place in their minds, the students are assigned (in class, homework and examinations) many standard exercises and problems related to cosets.

After the students have been through such an instructional treatment, quantitative and qualitative instruments are designed to determine the mental concepts they may have constructed and the mathematics they may have learned. The theoretical analysis points to questions researchers may ask in the process of data analysis and the results of this data analysis indicates both the extent to which the instruction has been effective and possible revisions in the genetic decomposition.

This way of doing research and curriculum development simultaneously emphasizes both theory and applications to teaching practice.

Refining the theory

As noted above, the theory helps us analyze data and our attempt to use the theory to explain the data can lead to changes in the theory. These changes can be of two kinds. Usually, the genetic decomposition in the original theoretical analysis is revised and refined as a result of the data. In rare cases, it may be necessary to enhance the overall theory. An important example of such a revision is the incorporation of the triad concept of Piaget and Garcia (1989) which is leading to a better understanding of the construction of schemas. This enhancement to the theory was introduced in Clark, et. al. (1997) where they report on students' understanding of the chain rule, and is being further elaborated upon in three current studies: sequences of numbers (Mathews, et. al., in preparation); the chain rule and its relation to composition of functions (Cottrill, 1999); and the relations between the graph of a function and properties of its first and second derivatives (Baker, et. al., submitted). In each of these studies, the understanding of schemas as described above was not adequate to provide a satisfactory explanation of the data and the introduction of the triad helped to elaborate a deeper understanding of schemas and provide better explanations of the data.

The triad mechanism consists in three stages, referred to as *Intra*, *Inter*, and *Trans*, in the development of the connections an individual can make between particular constructs within the schema, as well as the coherence of these connections. The Intra stage of schema development is characterized by a focus on individual actions, processes, and objects in isolation from other cognitive items of a similar nature. For example, in the function concept, an individual at the Intra level, would tend to focus on a single function and the various activities that he or she could perform with it. The

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Inter stage is characterized by the construction of relationships and transformations among these cognitive entities. At this stage, an individual may begin to group items together and even call them by the same name. In the case of functions, the individual might think about adding functions, composing them, etc. and even begin to think of all of these individual operations as instances of the same sort of activity: transformation of functions. Finally, at the Trans stage the individual constructs an implicit or explicit underlying structure through which the relationships developed in the Inter stage are understood and which gives the schema a coherence by which the individual can decide what is in the scope of the schema and what is not. For example, an individual at the Trans stage for the function concept could construct various systems of transformations of functions such as rings of functions, infinite dimensional vector spaces of functions, together with the operations included in such mathematical structures.

Applying the APOS Theory

Included with this paper is an annotated bibliography of research related to APOS Theory, its ongoing development and its use in specific research studies. This research concerns mathematical concepts such as: functions; various topics in abstract algebra including binary operations, groups, subgroups, cosets, normality and quotient groups; topics in discrete mathematics such as mathematical induction, permutations, symmetries, existential and universal quantifiers; topics in calculus including limits, the chain rule, graphical understanding of the derivative and infinite sequences of numbers; topics in statistics such as mean, standard deviation and the central limit theorem; elementary number theory topics such as place value in base n numbers, divisibility, multiples and conversion of numbers from one base to another; and fractions. In most of this work, the context for the studies are collegiate level mathematics topics and undergraduate students. In the case of the number theory studies, the researchers examine the understanding of pre-college mathematics concepts by college students preparing to be teachers. Finally, some studies such as that of fractions, show that the APOS Theory, developed for "advanced" mathematical thinking, is also a useful tool in studying students' understanding of more basic mathematical concepts.

The totality of this body of work, much of it done by RUMEC members involved in developing the theory, but an increasing amount done by individual researchers having no connection with RUMEC or the construction of the theory, suggests that APOS Theory is a tool that can be used objectively to explain student difficulties with a broad range of mathematical concepts and to suggest ways that students can learn these concepts. APOS Theory can point us towards pedagogical strategies that lead to marked improvement in student learning of complex or abstract mathematical concepts and students' use of these concepts to prove theorems, provide examples, and solve problems. Data supporting this assertion can be found in the papers listed in the bibliography.

Using the APOS Theory to develop a community of researchers

At this stage in the development of research in undergraduate mathematics education, there is neither a sufficiently large number of researchers nor enough graduate school programs to train new researchers. Other approaches, such as experienced and novice researchers working together in teams on specific research problems, need to be employed at least on a temporary basis. RUMEC is one example of a research community that has utilized this approach in training new researchers.

In addition, a specific theory can be used to unify and focus the work of such groups. The initial group of researchers in RUMEC, about 30 total, made a decision to focus their research work around the APOS Theory. This was not for the purpose of establishing dogma or creating a closed research community, but rather it was a decision based on current interests and needs of the group of researchers.

RUMEC was formed by a combination of established and beginning researchers in mathematics education. Thus one important role of RUMEC was the mentoring of these new researchers. Having a single theoretical perspective in which the work of RUMEC was initially grounded was beneficial for those just beginning in this area. At the meetings of RUMEC, discussions could focus not only on the details of the individual projects as they developed, but also on the general theory underlying all of the work. In addition, the group's general interest in this theory and frequent discussions about it in the context of active research projects has led to growth in the theory itself. This was the case, for example, in the development of the triad as a tool for understanding schemas.

As the work of this group matures, individuals are beginning to use other theoretical perspectives and other modes of doing research.

Summary

In this paper, we have mentioned six ways in which a theory can contribute to research and we suggest that this list can be used as criteria for evaluating a theory. We have described how one such perspective, APOS Theory is being used, in an organized way, by members of RUMEC and others to conduct research and develop curriculum. We have shown how observing students' success in making or not making mental constructions proposed by the theory and using such observations to analyze data can organize our thinking about learning mathematical concepts, provide explanations of student difficulties and predict success or failure in understanding a mathematical concept. There is a wide range of mathematical concepts to which APOS Theory can and has been applied and this theory is used as a language for communication of ideas about learning. We have also seen how the theory is grounded in data, and has been used as a vehicle for building a community of researchers. Yet its use is not restricted to members of that community. Finally, we provide an annotated bibliography which presents further details about this theory and its use in research in undergraduate mathematics education.

An Annotated Bibliography of works
which develop or utilize APOS Theory

I. Arnon. Teaching fractions in elementary school using the software "Fractions as Equivalence Classes" of the Centre for Educational Technology, *The Ninth Annual Conference for Computers in Education, The Israeli Organization for Computers in Education, Book of Abstracts*, Tel-Aviv, Israel, p. 48, 1992. (In Hebrew).

I. Arnon, R. Nirenburg and M. Sukenik. Teaching decimal numbers using concrete objects, *The Second Conference of the Association for the Advancement of the Mathematical Education in Israel, Book of Abstracts*, Jerusalem, Israel, p. 19, 1995. (In Hebrew).

I. Arnon. Refining the use of concrete objects for teaching mathematics to children at the age of concrete operations, *The Third Conference of the Association for the Advancement of the Mathematical Education in Israel, Book of Abstracts*, Jerusalem, Israel, p. 69, 1996. (In Hebrew).

I. Arnon. In the mind's eye: How children develop mathematical concepts – extending Piaget's theory. Doctoral dissertation, School of Education, Haifa University, 1998a.

I. Arnon. Similar stages in the developments of the concept of rational number and the concept of decimal number, and possible relations between their developments, *The Fifth Conference of the Association for the Advancement of the Mathematical Education in Israel, Book of Abstracts.* Be'er-Tuvia, Israel, p. 42, 1998b. (In Hebrew).

> The studies by Arnon and her colleagues listed above deal with the development of mathematical concepts by elementary school children. Having created a framework that combines APOS theory, Nesher's theory on Learning Systems, and Yerushalmy's ideas of multi-representation, she investigates the introduction of mathematical concepts as *concrete actions* versus their introduction as *concrete objects***.** She establishes developmental paths for certain fraction-concepts. She finds that students to whom the fractions were introduced as concrete actions progressed better along these paths than students to whom the fractions were introduced as concrete objects. In addition, the findings establish the following stage in the development of concrete actions into abstract objects: after abandoning the concrete materials, and before achieving abstract levels, children perform the concrete actions in their imagination. This corresponds to the interiorization of APOS theory.

M. Artigue, Enseñanza y aprendizaje del análisis elemental: ¿qué se puede aprender de las investigaciones didácticas y los cambios curriculares? *Revista Latinoamericana de Investigación en Matiemática Educativa*, **1**, 1, 40-55, 1998.

> In the first part of this paper, the author discusses a number of student difficulties and tries to explain them using various theories of learning including APOS Theory. Students' unwillingness to accept that 0.999… is equal to 1 is explained, for example, by interpreting the former as a process, the latter as an object so that the two cannot be seen as equal until the student is able to encapsulate the process which is a general difficulty. In the second part of the paper, the author discusses the measures that have been taken in France during the 20th Century to overcome these difficulties.

M. Asiala, A. Brown, D. DeVries, E. Dubinsky, D. Mathews and K. Thomas. A framework for research and curriculum development in undergraduate mathematics education, *Research in Collegiate Mathematics Education II, CBMS Issues in Mathematics Education*, **6**, 1-32, 1996.

> The authors detail a research framework with three components and give examples of its application. The framework utilizes qualitative methods for research and is based on a very specific theoretical perspective that was developed through attempts to understand the ideas of Piaget concerning reflective abstraction and reconstruct them in the context of college level mathematics. For the first component, the theoretical analysis, the authors present the APOS theory. For the second component, the authors describe specific instructional treatments, including the ACE teaching cycle (activities, class discussion, and exercises), cooperative learning, and the use of the programming language ISETL. The final component consists of data collection and analysis.

M. Asiala, A. Brown, J. Kleiman and D. Mathews. The development of students' understanding of permutations and symmetries, *International Journal of Computers for Mathematical Learning*, **3**, 13-43, 1998.

> The authors examine how abstract algebra students might come to understand permutations of a finite set and symmetries of a regular polygon. They give initial theoretical analyses of what it could mean to understand permutations and symmetries, expressed in terms of APOS. They describe an instructional approach designed to help foster the formation of mental constructions postulated by the theoretical analysis, and discuss the results of interviews and performance on examinations. These results suggest that the pedagogical approach was reasonably effective in helping students develop strong conceptions of permutations and symmetries. Based on the data collected as part of this study, the authors propose revised epistemological analyses of permutations and symmetries and give pedagogical suggestions.

M. Asiala, J. Cottrill, E. Dubinsky and K. Schwingendorf. The development of student's graphical understanding of the derivative, *Journal of Mathematical Behavior*, **16**(4), 399-431, 1997.

> In this study the authors explore calculus students' graphical understanding of a function and its derivative. An initial theoretical analysis of the cognitive constructions that might be necessary for this understanding is given in terms of APOS. An instructional treatment designed to help foster the formation of these mental constructions is described, and results of interviews, conducted after the implementation of the instructional treatment, are discussed. Based on the data collected as part of this study, a revised epistemological analysis for the graphical understanding of the derivative is proposed. Comparative data also suggest that students who had the instructional treatment based on the theoretical analysis may have more success in developing a graphical understanding of a function and its derivative than students from traditional courses.

M. Asiala, E. Dubinsky, D. Mathews, S. Morics and A. Oktac. Student understanding of cosets, normality and quotient groups, *Journal of Mathematical Behavior*, **16**(3), 241-309, 1997.

> Using an initial epistemological analysis from Dubinsky, Dautermann, Leron and Zazkis (1994), the authors determine the extent to which the APOS perspective explains students' mental constructions of the concepts of cosets, normality and quotient groups, evaluate the

effectiveness of instructional treatments developed to foster students' mental constructions, and compare the performance of students receiving this instructional treatment with those completing a traditional course.

T. Ayers, G. Davis, E. Dubinsky and P. Lewin. Computer experiences in the teaching of composition of functions, *Journal for Research in Mathematics Education*, **19**(3), 246-259, 1988.

Students from two sections of a college mathematics lab (n=13) who were given computer experiences to help induce reflective abstraction scored higher on a test of their understanding of functions and compositions than students from another section (n=17) who were taught according to traditional methods. The comparison was based on questions intended to indicate whether reflective abstraction had taken place.

B. Baker, L. Cooley and M. Trigueros. The schema triad – A calculus example. Submitted for publication.

In this paper the authors report on the ways students try to solve a non-routine mathematical problem which involves graphing a function given certain properties such as particular limits associated with the function, continuity, and first and second derivative information. In order to analyze the interview data the authors extend the theory of schema development within the APOS theory. They found that students had to deal with two different schemas and that, for a particular student, those schemas could be at different stages of development. The first schema relates to the real line and the handling of information given in overlapping intervals; the second is the schema for the various properties of the function. Building on work of Piaget and Garcia as well as Clark, Cordero, et. al. (1997), the authors develop a double triad theoretical framework in terms of the interaction between these two schemas and classify the students in these two triad levels. They find student responses at each of the nine possible levels except one: the trans property – intra interval level. In addition to the theoretical development, the authors pinpoint specific difficulties students had with this problem: understanding the graphical meaning of the second derivative, imagining a continuous curve that has a cusp, drawing points of inflection, and relating continuity to differentiability.

D. Breidenbach, E. Dubinsky, J. Hawks and D. Nichols. Development of the process conception of function, *Educational Studies in Mathematics*, **23**, 247-285, 1992.

The authors argue that APOS theory, and how it applies to the concept of function, points to an instructional treatment, using computers, that results in substantial improvements in the understanding of function for many students. The students appear to develop a process conception of function and are able to use it to perform certain mathematical tasks.

A. Brown, D. DeVries, E. Dubinsky and K. Thomas. Learning binary operations, groups, and subgroups, *Journal of Mathematical Behavior*, **16**(3), 187-239, 1997.

The authors examine how abstract algebra students might come to understand binary operations, groups, and subgroups. Using APOS theory, they give preliminary analyses of what it could mean to understand these topics. They describe an instructional treatment designed to help foster the formation of mental constructions postulated by the theoretical analysis, and discuss the results of interviews and exam performance which suggest that the

instruction was successful. Based on the data collected, they propose revised epistemological analyses of these topics, and give some further pedagogical suggestions.

A. Brown, K. Thomas and G. Tolias. The nature and development of preservice elementary teachers' understanding of divisibility. In preparation.

The authors report on an examination of prospective elementary teachers' understanding of the concept of multiples, with a particular focus on the least common multiple. Students' understanding is examined using the APOS theory.

M. Brust. Solucion de Ecuaciones de segundo Grado. B.Sc. Instituto Tecnologico Autonomo de Mexico, 1997.

APOS is used to make an initial decomposition of what it might mean to understand the concept of solution of a second degree equation and to design and subsequently analyze interviews. Fifteen students ages 15-16 were interviewed to analyze their understanding of what a solution to a second degree equation means, their understanding of parameters included in an equation or in the general quadratic formula used for solving such equations, their ability to manipulate equations, their flexibility in choosing a solution method depending of the particular equation, their ability to interpret graphically the meaning of the solution and their ability to symbolize verbal problems which yield second degree equations. The results show that students do not clearly understand the role of the variable and of parameters in the equations or the quadratic formula. They have difficulties interpreting the meaning of the solution graphically. Students also show a strong tendency to use arithmetic methods and to generalize the actions they follow when solving first order equations. Almost all of the students did not believe that different methods of solution were in fact equivalent. This implies that most of the students were at an action level of understanding, although a few were at a process level.

M. Carlson. A cross-sectional investigation of the development of the function concept, *Research in Collegiate Mathematics Education III*, *CBMS Issues in Mathematics Education*, **7**, 114-162, 1998.

In this study the author investigates students' development of the function concept as they progress through undergraduate mathematics. An exam measuring students' understandings of major aspects of the function concept was developed and administered to students who had just received A's in college algebra, second-semester honors calculus, or first-year graduate mathematics courses. Follow-up interviews were conducted with five students from each of these groups. The data analysis procedure takes into account APOS theory as well as other frameworks researchers have used to classify students' conceptual views of function. The author reaches a number of conclusions, including agreement with Breidenbach, et al. (1992) that students' understanding of functions was improved as a result of engaging in construction activities.

G. Carmona. The concept of tangent and its relationship with the concept of derivative. Doctoral thesis, Instituto Tecnologico Autonomo de Mexico, 1996.

Selected students who had finished two calculus courses, or who were at the end of their studies were given a questionnaire and interviewed. APOS decompositions of the concepts of

tangent and derivative based on the results of Tostado (1995) were used. It was found that some of the students' conceptions are very resistant to change even when they are taught using computers or other non-traditional methods. Moreover, even if the students do change, after the calculus courses their original conceptions tend to reappear when they loose contact with formal mathematics.

J. Clark, F. Cordero, J. Cottrill, B. Czarnocha, D. J. DeVries, D. St. John, G. Tolias and D. Vidakovic. Constructing a schema: The case of the chain rule, *Journal of Mathematical Behavior*, **16**(4), 345-364, 1997.

> Based on an initial description (genetic decomposition) of how the chain rule concept may be learned, an attempt to interpret student interview data using APOS was made. The insufficiency of this alone led to an extension of the APOS theory to include a theory of schema development based on ideas of Piaget and Garcia. The Piagetian triad is suggested as a mechanism for describing schema development in general, and the chain rule is used as an example. The triad of the intra-, inter- and trans- levels of schema development provides the structure for interpreting the students' understanding of the chain rule and classifying their responses to interview questions about the chain rule. The results of this data analysis allowed for a revised epistemological analysis of the chain rule.

J. Clark, D. DeVries, G. Litman, M. Meletiou, S. Morics, K. Schwingendorf and D. Vidakovic. A story of progress: Research in undergraduate mathematics education. Submitted for publication.

> To make the point that when a group of researchers adheres to a particular framework and theoretical perspective over a period of time, considerable progress can be made both in understanding how students learn mathematics and in building pedagogy on that understanding, the authors report on the collective works of a group of researchers who have adopted APOS as the basis for research studies of college students' cognitive development and understandings of mathematical concepts. This paper provides a brief overview of APOS theory followed by a sample discrete mathematics lesson on learning (single-level) quantification. This lesson is offered as an example of how research rooted in the APOS perspective has led to pedagogical strategies. The paper then summarizes some of the research, the instructional materials, and the methods that have been developed over the past decade using this framework and theoretical perspective. The research described is in the areas of student understandings of concepts in pre-calculus, calculus, abstract algebra, statistics, and discrete mathematics.

J. Clark, C. Hemenway, D. St. John, G. Tolias and R. Vakil. Student attitudes toward Abstract Algebra, *Primus*, to appear.

> The authors report on one study of a research and curriculum development program in abstract algebra. The instructional treatment is based on APOS theory and places special emphasis on computer programming activities and cooperative learning. Students from both this and more traditional courses were interviewed about their impressions of the course and abstract algebra in general. Their responses favored the computer/cooperative learning approach in many ways, even though the content of this course was at least as rigorous and demanding for them as that of the more traditional courses.

J. Clark and D. Mathews. Successful students' conceptions of mean, standard deviation and the Central Limit Theorem. In preparation.

> The authors present analyses based on APOS of audio-taped clinical interviews with college freshmen immediately after they completed an elementary statistics course and obtained a grade of "A". The authors find that APOS is a useful way of describing students' understanding of mean, standard deviation, and the Central Limit Theorem. In addition, they conclude that traditional instruction in statistics does not help students make the appropriate mental constructions. In particular, traditional instruction seems to inhibit students from moving from a process to an object conception of standard deviation, and that it is very difficult for students to move beyond a strong process image of standard deviation.

F. Cordero. Entendimiento de los conceptos del Análisis y Cálculo. Las construcciones mentales como un marco epistemológico, en (R. Farfán, ed.) *Actas de la Undécima Reunión Latinoamericana de Matemática Educativa*, Grupo Editorial Iberoamérica, primera edición, 38-41, 1997.

> The author investigates the meaning of mental constructions when a theoretical perspective puts together understanding, epistemological frameworks, and functional aspects of mathematical knowledge. He concludes that the notion of development of schema in APOS theory plays a very important role.

F. Cordero y M. Solís. Actos visuales y analíticos en el entendimiento de la ecuaciones diferenciales lineales, en (R. Farfán, ed.) *Actas de la Undécima Reunión Latinoamericana de Matemática Educativa*, Grupo Editorial Iberoamérica, primera edición, 69-73, 1997.

> The authors report on a research project about understanding linear differential equations using analytic and visual acts based on mental constructions as described in APOS theory.

F. Cordero. El entendimiento de algunas categorías del conocimiento del cálculo y análisis: el caso de comportamiento tendencial de las funciones, *Revista Latinoamericana de Investigación en Matemática Educativa*, Número 1, 56-74, 1998.

> In the school-teaching context, the author encountered an argument given by students on the subject of graphs of functions. He calls this argument the "tendencial behavior of functions" because of its nature. The author shows some constructions of this argument as done by students and analyzes their data using a version of APOS theory.

J. Cottrill. Students' understanding of the concept of chain rule in first year calculus and the relation to their understanding of composition of functions. Doctoral dissertation, Purdue University. (Completion expected 1999.)

> This is a follow-up study to Clark, Cordero, et. al (1997). The author finds that the triad mechanism describes the observations of student behaviors and can be used to develop instruction to help students make certain mental constructions. It presents more detailed descriptions of the intra-, inter-, and trans- levels of the development of the chain rule schema than were given in Clark, Cordero, et. al (1997).

J. Cottrill, E. Dubinsky, D. Nichols, K. Schwingendorf, K. Thomas and D. Vidakovic.   Understanding the limit concept: Beginning with a coordinated process schema, *Journal of Mathematical Behavior*, **15**(2), 167-192, 1996.

> The authors suggest a new variation of the dichotomy between dynamic or process conceptions of limit and static or formal conceptions. They also propose explanations of why these conceptions are so difficult for students to construct.

E. Dubinsky and P. Lewin. Reflective abstraction and mathematics education: The genetic decomposition of induction and compactness, *Journal of Mathematical Behavior*, **5**, 55-92, 1986.

> The authors formulate the beginning of a theory of learning abstract mathematical concepts at the post-secondary level (a precursor to APOS theory) by interpreting Piaget's epistemology, focusing on the model of equilibration and the concept of reflective abstraction. They then use the concepts of mathematical induction and compactness to elucidate the theoretical ideas and show that it is possible to arrive at coherent genetic decompositions of fairly sophisticated concepts.

E. Dubinsky. Teaching mathematical induction I. *Journal of Mathematical Behavior*, **6**(1), 305-317, 1987.

> A prototype version of a novel approach to teaching mathematical induction was used in a small class. The instructional treatment was based on an early version of the APOS theory of learning abstract mathematical concepts in which the learner uses reflective abstraction to construct new schemas out of old ones in a hierarchy that ultimately reaches the desired concept. The treatment uses certain computer experiences in an attempt to induce the student to make the appropriate reflective abstraction. The method is seen to be reasonably effective and several areas of possible improvement are indicated.

E. Dubinsky, F. Elterman and C. Gong. The student's construction of quantification, *For the Learning of Mathematics*, **8**(2), 44-51, 1988.

> In this paper, the authors detail a proposed genetic decomposition for the concept of quantification. The observations are taken from an informal study of a Discrete Mathematics class where quantification was a major topic and the instructional treatment used computer experiences with SETL, the programming language on which ISETL is based.

E. Dubinsky. On teaching mathematical induction II. *Journal of Mathematical Behavior*, **8**, 285-304, 1989.

> This paper is a continuation of Dubinsky and Lewin (1986) and Dubinsky (1987). Here the author details two classroom experiments in which a theoretically-based instructional approach using computer experiences with SETL and ISETL was implemented. Students seem to develop a more positive attitude toward making induction proofs. They are totally successful in solving straight-forward problems. When presented with more difficult, unfamiliar problems, they tend to set up most problems correctly and it is usually clear that the students know how to use induction and intend to do so, although some difficulties with specific proofs persist.

E. Dubinsky. Reflective abstraction in advanced mathematical thinking, in (D. Tall, ed.) *Advanced Mathematical Thinking*, Dordrecht: Kluwer, 95-126, 1991a.

> The author makes the case that the concept of reflective abstraction can be a powerful tool in the study of advanced mathematical thinking, that it can provide a theoretical basis that supports and contributes to our understanding of what this thinking is, and suggests how we can help students develop the ability to engage in it.

E. Dubinsky. The constructive aspects of reflective abstraction in advanced mathematics, in (L. P. Steffe, ed.) *Epistemological Foundations of Mathematical Experiences*, New York: Springer-Verlag, 1991b.

> The author presents a brief discussion of a developing theory of mathematical knowledge and its acquisition. He also describes specific methods of construction that he has observed in students. He presents an analysis of induction, quantification, and function that have been studied using this point of view.

E. Dubinsky. A learning theory approach to calculus, in (Z. Karian, ed.) *Symbolic computation in undergraduate mathematics education*, MAA Notes, **24**, Mathematical Association of America, 48-55, 1992.

> The author outlines the APOS theory of how people can learn mathematical concepts. He then discusses some of the choices about teaching that seem to follow from the beliefs about learning to which this theory has led him. In particular, he discusses how computers can be used in teaching and learning.

E. Dubinsky and G. Harel. The nature of the process conception of function, in (G. Harel and E. Dubinsky, eds.) *The concept of function: Aspects of epistemology and pedagogy*, MAA Notes, **25**, Mathematical Association of America, 85-106, 1992.

> The authors examine interviews with 13 students who have gone through an instructional treatment based on APOS theory and which involved ISETL programming activities to see how far beyond an action conception and how much into a process conception each student was at the end of the instruction. The authors find that the process conception of function is very complex and examine the data through a number of facets: restrictions students possess about what a function is, the severity of the restrictions, students' ability to construct a process when none is explicit in the situation, and their confusion with one-to-one.

E. Dubinsky. A theory and practice of learning college mathematics, in (A. Schoenfeld, ed.) *Mathematical Thinking and Problem Solving*. Hillsdale: Erlbaum, 221-243, 1994.

> The author examines two dichotomies and looks at ways to build syntheses between two apparently disparate notions. The two dichotomies examined are that of research and development, and beliefs and choices. As part of the examination, the APOS theoretical perspective involving the mental construction of processes and objects is presented.

E. Dubinsky, J. Dautermann, U. Leron and R. Zazkis. On learning fundamental concepts of group theory, *Educational Studies in Mathematics*, **27**(3), 267-305, 1994.

    The authors present one of the first systematic investigations of students' construction of the concepts of group, subgroup, coset, normality and quotient group. Using APOS theory, the authors make general observations about learning these specific topics, the complex nature of "understanding", and the role of errors and misconceptions.

E. Dubinsky. ISETL: A programming language for learning mathematics, *Communications on Pure and Applied Mathematics*, **48**, 1-25, 1995.

    The author gives a brief history of the development of a pedagogical strategy for helping students learn mathematical concepts at the post-secondary level. The method uses ISETL to implement instruction designed on the basis of APOS theory. ISETL is described in some detail and examples are given of the use of this pedagogy in abstract algebra, calculus, and mathematical induction.

E. Dubinsky, On learning quantification, *Journal of Computers in Mathematics and Science Teaching*, **16**(2/3), 335-362, 1997.

    In this study the author examines students' learning of universal and existential quantification. The instruction in the course was based on the theoretical analysis of quantification found in Dubinsky, Elterman and Gong (1988) and was designed to assist students to make effective mathematical constructions in their mind by making these constructions on a computer using ISETL. Results from written questions suggest that when the pedagogical approach described is used, students can develop some understanding of quantification and the ability to work with it, even when the particular problems they are given are difficult.

J. Kuhn, G. McCabe and K. Schwingendorf. A longitudinal study of the C^{4}L reform program: Comparisons of C^{4}L and traditional students. Submitted for publication.

    The authors present results of a statistical comparison between 205 students who took the course Calculus, Concepts, Computers and Cooperative Learning (a reform course designed using APOS theory) and 4431 students who took a traditional calculus course at Purdue University. The data consists of grades and the numbers of calculus and non-calculus mathematics courses taken by each student. The reform course students earned higher grades in calculus courses, were as adequately prepared for math courses beyond calculus as well as all other academic courses, took more calculus courses, and took about the same number of non-calculus mathematics courses as the traditionally taught students.

D. Lozano. El Concepto de Variable: Evolución a lo largo de la Instrucción Matemática. B.Sc. Thesis, Instituto Tecnologico Autonomo de Mexico, 1998.

    The author starts from a decomposition of the different uses of variable in elementary algebra previously developed by Ursini and Trigueros. The main objective was to study how the understanding of variable evolves through schooling and thus the study involved students aged 12-18 years. A questionnaire and interviews were used to analyze the ways students handle and conceptualize the different uses of variable – including variable as an unknown, variable as

a general number and variable in a functional relationship. Results show that students' conceptions of variable do not improve substantially as more algebra courses are taken, and that students' difficulties are not of a cognitive or epistemological nature, but that they are a consequence of current didactical approaches.

D. Mathews, M. A. McDonald and K. Strobel. Understanding sequences: A tale of two objects. In preparation.

APOS was used to examine students' cognitive construction of the concept of sequence. The authors show that students tend to construct two distinct cognitive objects and refer to both as a sequence. One construction, which the authors call SEQLIST, is what one might understand as a listing representation of a sequence. The other, which they call SEQFUNC, is what one might interpret as a functional representation of a sequence. As the connections between these two entities becomes stronger, and the students reflect on these connections, they begin to understand sequence as a single cognitive entity and SEQLIST and SEQFUNC as mathematical representations of this entity. In this paper the authors detail the construction of SEQLIST and SEQFUNC by the students, and characterize the connections between them through the triad model of schema development introduced by Clark, Cordero, et. al. (1997).

M. A. Tostado. Derivative and tangent: A longitudinal study. Doctoral thesis, Instituto Tecnologico Autonomo de Mexico, 1995.

In this thesis the author explores students' conceptions about tangent and derivative in a graphical context. A questionnaire and interviews were used to gather data from students at different grades at the university, enrolled in different majors. APOS was used to generate a genetic decomposition of both concepts and it was also used to analyze students' responses to the questionnaire. The author tried to see if the understanding of students improves when they take more mathematics courses and if the understanding is different according to their major. What was found is that there is an improvement in student's understanding while they are taking calculus courses but they forget very quickly what they seemed to have learned. Only students in mathematics majors do better at the end of their studies than at the beginning; the others regress almost to their original conceptions.

M. Trigueros, S. Ursini, R. Quintero, and A. Reyes. Students' approaches to different uses of variable. *Proceedings of the XIX PME Conference*, 1995.

M. Trigueros, S. Ursini and A. Reyes. College students' conceptions of variable. *Proceedings of the XX PME International Conference*, Spain, 1996.

M. Trigueros and S. Ursini. Understanding of different uses of variable: A study with starting college students. *Proceedings of the XXI PME International Conference*, Finland, 1997.

M. Trigueros and S. Ursini. Starting college students' difficulties in working with different uses of variable. Submitted for publication.

In the above studies by Trigueros, Ursini, and colleagues, the authors examine college students' understanding of variable. In particular, they examine students' ability to interpret and use variables as unknown, as general number, and as variables in simple functional relationships.

The results are all based on a detailed analysis of responses given by 164 starting college students to a questionnaire of 65 open ended items. The results show the persistence of misconceptions and approaches characteristic of beginning algebra students in lower school levels. Evidence suggests on the one hand that students' are anchored at an action level, in which their responses seem to be a reaction to key signs present in an expression (quadratic exponent, equal sign). On the other hand, there is a tendency for students to rely on and to apply memorized rules without determining whether they are pertinent to the given situation. Finally, most students have difficulties shifting to levels of abstraction at which variables are conceived as objects.

D. Vidakovic. Cooperative learning: Differences between group and individual processes of construction of the concept of inverse function. Unpublished doctoral dissertation, Purdue University, 1993.

The study was conducted with five groups of students working together on some learning activities, and five individuals working alone on the same tasks. The activities were related to the idea of function, in particular the inverse and composition of a function. The goal of the study was to learn more about how these concepts can be learned, and hence taught, as well as to investigate the differences between group and individual mental constructions of the concepts. The APOS (action-process-object-schema) theoretical framework was used as a guideline in designing the study and analyzing and interpreting the data. As a result of the study, a genetic decomposition of the concept of inverse function was deduced and an instructional treatment was developed. Further, as a result of this inductive empirical investigation, recommendations in choosing group work over individual problem-solving have been made.

M. Wahlberg. The effects of writing assignments on second-semester calculus students' understanding of the limit concept. Submitted for publication.

Over a semester, the treatment group (n=37) completed four problem sets plus six limit-oriented writing assignments that replaced six problem sets. The control group (n=34) completed ten problem sets. A subset of the treatment group (n=5) were interviewed three times to ascertain current limit understanding as well as cognitive growth. APOS theory was used to analyze interview transcripts and writing assignments. A quantitative analysis was performed as well, to compare the two groups' performance on three limit-oriented final examination problems. The treatment group showed cognitive growth and outperformed the control group on the limit problems on the final examination.

R. Zazkis and H. Khoury. To the right of the decimal point: Preservice teachers' concepts of place value and multidigit structures. *Research in Collegiate Mathematics Education I*, *CBMS Issues in Mathematics Education*, **4**, 195-224, 1994.

Rational numbers represented in bases other than ten and referred to as "non-decimals" are used to discuss preservice teachers' ideas about decimal fractions and place value representation. The analysis highlights possible pitfalls and conceptual difficulties that are not apparent without stepping away from a decimal representation.

R. Zazkis and S. Campbell. Divisibility and multiplicative structure of natural numbers: Preservice teachers' understanding. *Journal for Research in Mathematics Education*, **27**(5), 540-563, 1996.

> Preservice elementary school teachers' understanding of divisibility in connection with division, multiplication and prime decomposition is analyzed. Results indicate pervasive dispositions toward procedural attachments, even when some degree of conceptual understanding was evident.

R. Zazkis and C. Gunn. Sets, subsets and the empty set: Students' constructions and mathematical conventions. *Journal of Computers in Mathematics and Science Teaching*, **16**(1), 133-169, 1997.

> Preservice elementary school teachers' understanding of basic concepts of Set Theory – set, set element, cardinality, subset and the empty set – is analyzed, following an experiment in ISETL. Analysis of the data is based on the action-process-object developmental framework and special attention is given to students' difficulties with the idea of a set as a set of elements and the idea of the empty set.