p-group Camina pairs

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  1. For every element $g \in G \setminus N$, the element g is conjugate to all of gN.
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  4. $|C_G(g)| = |C_{G/N}(gN)|$ for all $g \in G \setminus N$.
- A pair (G, N) is a *Camina pair* if it satisfies the above conditions.

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Today, we consider the case where $N = Z(G)$.

Basics

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Motivation:

Theorem (Macdonald)

Let G be a Camina group of nilpotence class 2. Then $|Z(G)|^2 \leq |G : Z(G)|$.

Basics

Lemma

Let $(G, Z(G))$ be a Camina pair. Every character in $\mathrm{Irr}(G \mid Z(G))$ is fully ramified with respect to $G/Z(G)$. In particular, $|G : Z(G)|$ is a square.

Exponent

Our first result:

Theorem

Let $(G, Z(G))$ be a Camina pair where G is a p-group. If $G/Z(G)$ has exponent p^n with $n \geq 1$, then $|Z(G)|^n p^n \leq |G : Z(G)|$. In particular, $|Z(G)|^n < |G : Z(G)|$.

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When $p = 2$, this yields:

Corollary

Let $(G, Z(G))$ be a Camina pair where G is a 2-group. Then $|Z(G)|^2 \leq |G : Z(G)|$. Furthermore, if equality holds, then G is a Camina group.

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Henceforth, we may assume that p is odd.

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Let $(G, Z(G))$ be a Camina pair. If G is a p-group and the exponent of $G/Z(G)$ is not p, then $|Z(G)| < |G : Z(G)|^{1/2}$.

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From now on, we assume that $G/Z(G)$ has exponent p.

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When $(G, Z(G))$ is a Camina pair and $Z(G) < G'$, then we also obtain a bound for $|Z(G)|$ in terms of $|G' : Z(G)|$.

Theorem

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Theorem

Let $(G, Z(G))$ be a Camina pair. Then $|Z(G)| < |G : Z(G)|^{3/4}$.

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Theorem

Let $(G, Z(G))$ be a Camina pair with $Z(G) < G'$. Then either $|Z(G)| \leq |G:Z(G)|^{1/2}$ or $|Z(G)|p^4 \leq |G:Z(G)|$.

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In particular, if $|G:Z(G)| \leq p^8$, then $|Z(G)| \leq |G:Z(G)|^{1/2}$.

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- Both $G/Z(G)$ and $Z(G)$ have exponent p.
- $Z(G)$ has order p^{2k}.
- $Z(G) = [G', G]$.
- $Z_2(G)$ is an abelian group of order p^{3k+1}.
  Note: $Z_2(G)/Z(G) = Z(G/Z(G))$.

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- Either $Z_2(G) = G'$ or $|Z_2(G) : G'| = p$.
- If $g \in G \setminus Z_2(G)$, then $C_G(g) = \langle g, Z(G) \rangle$.

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- If $h \in D(g) \setminus Z_2(G)$, then $D(g) = D(h)$.
- $G \setminus Z_2(G)$ is partitioned by the sets $D(g) \setminus Z_2(G)$ as g runs over the elements in $G \setminus Z_2(G)$.

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- If $h \in D(g) \setminus Z_2(G)$, then $C_{D(g)}(h) = \langle h, Z(G) \rangle$, and so $|D(g) : C_{D(g)}(h)| = p^{2k} = |D(g)'|$.

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- This implies that $\mathrm{cl}_{D(g)}(h) = hZ(G)$.
- $D(g)$ is special, and in fact, $Z(D(g)) = Z(G)$.

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Lemma

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