Induction and Restriction of Characters and Hall subgroups

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We see that in general, we obtain a relatively weak conclusion.

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We present our main result regarding induction.

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Theorem

Let $H \leq G$, and write $N = \mathrm{core}_G(H)$. Suppose that H/N is a Hall π-subgroup of G/N for some set π of primes, and assume that G/N is π-separable. Let $\chi \in \mathrm{Irr}(G)$, and suppose that $\alpha^G = \chi$ for each irreducible constituent α of χ_H. Then $\chi = \beta^G$ for each irreducible constituent β of χ_N.

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The converse for this theorem is almost trivial.

To see this, suppose $N \leq H \leq G$ and $\chi \in \mathrm{Irr}(G)$ has the property that $\chi = \beta^G$ for each irreducible constituent β of χ_N.

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Next, we consider restriction.

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Let $H \leq G$ and write $N = \mathrm{core}_G(H)$. Suppose that H/N is a Hall π-subgroup of G/N for some set π of primes, and assume that G/N is π-separable. Let $\alpha \in \mathrm{Irr}(H)$, and suppose that $\chi_H = \alpha$ for each irreducible constituent χ of α^G. Then $\alpha = \beta^H$ for some character β of N. Also, the Hall π'-subgroups of G/N are abelian.

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The conclusion in the first theorem that χ is induced from N and in the second theorem that α is induced from N definitely do not hold for arbitrary subgroups H of a group G.

In fact, there are counterexamples in which G is a 2-group. (In the induction case, there is a counter example with $|G| = 32$.)

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In this situation, $\sum (e_\chi)^2 = |G : H|$, where χ runs over the irreducible constituents of α^G.

Now, suppose that H is a Hall π-subgroup of a π-separable group G and let $\alpha \in \mathrm{Irr}(H)$.

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Theorem

Let G be π-separable for some set π of primes, and let $H \leq G$ be a Hall π-subgroup. Suppose that $\alpha \in \mathrm{Irr}(H)$ and that $(\alpha^G)_H$ is a multiple of α. Then the Hall π'-subgroups of G are abelian if and only if all associated coefficients are π-numbers.

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Also, we can weaken the assumption that $(\alpha^G)_H$ is a multiple of α and assume only that there exists some character of G whose restriction to H is a multiple of α.

In fact, we can weaken this still further and assume only that α_N is G-invariant.

Introduction

Theorem

Let H be a Hall π-subgroup of G for some set π of primes, and write $N = \mathrm{core}_G(H)$. Let $\alpha \in \mathrm{Irr}(H)$, and suppose that $\alpha = \beta^H$ for some character β of N. Suppose also that α_N is G-invariant. Then G has a Hall π'-subgroup K stabilizing β, and all such Hall subgroups are conjugate via elements of N. Also, $(\alpha^G)_H$ is multiple of α so the associated coefficients are defined, and they are exactly the irreducible character degrees (counting multiplicities) of K, and thus K is abelian if and only if all associated coefficients equal 1.

Introduction

We note that the main idea in the proof of the last two theorems is the generalization of the Gluck-Wolf theorem by Manz and Staszewski which states that if G is a π-separable group, N is a normal subgroup of G, and there is a character $\theta \in \mathrm{Irr}(N)$ so that no prime in π divides $\chi(1)/\theta(1)$ for all $\chi \in \mathrm{Irr}(G \mid \theta)$, then G/N has abelian Hall π-subgroups.

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We also need the following technical lemma which is easy to prove:

Introduction

Lemma

Let $H \leq G$*, and write* $N = \mathrm{core}_G(H)$*. Also, let* $\alpha \in \mathrm{Irr}(H)$ *and suppose that* $(\alpha^G)_H$ *is a multiple of* α*. Let* T *be the stabilizer in* G *of some irreducible constituent* β *of* α_N*, write* $S = T \cap H$ *and let* $\gamma \in \mathrm{Irr}(S)$ *be the Clifford correspondent of* α *with respect to* β*. The following then hold:*

Introduction

Lemma

1. $G = TH$, *and thus* $|G : T| = |H : S|$ *and* $|G : H| = |T : S|$.
2. $(\gamma^T)_S$ *is a multiple of* γ.
3. *Induction defines a bijection from the set* $\mathcal{Y}$ *of irreducible constituents of* γ^T *onto the set* $\mathcal{X}$ *of irreducible constituents of* α^G.
4. *If* $\eta \in \mathcal{Y}$ *and* $\chi = \eta^G \in \mathcal{X}$, *then* $\chi(1)/\alpha(1) = \eta(1)/\gamma(1)$, *and thus the associated coefficients for* α *and for* γ *are identical, counting multiplicities.*

Applications

A conjecture of G. Navarro asserts that if G is π-separable and H and K are respectively a Hall π-subgroup and a Hall π'-subgroup of G, then G has an irreducible character of degree at least the product of the maximal irreducible character degrees of H and K.

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Theorem

Let G be a group and let H and K be a Hall π-subgroup and Hall π'-subgroup of G, respectively. If $\max\{\chi(1) \mid \chi \in \mathrm{Irr}(G)\} = \max\{\theta(1) \mid \theta \in \mathrm{Irr}(H)\}$, then K is abelian.

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By our second theorem, K is abelian.

Applications

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In our second application, we consider *p*-Brauer characters of *p*-solvable groups.

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Suppose that $\varphi \in \mathrm{IBr}(G)$, where G is *p*-solvable.

If χ is an ordinary character of G, then we write χ° for the restriction of χ to the *p*-regular elements of G.

We say that χ is a lift of φ if $\chi^\circ = \varphi$.

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We ask when it happens that this maximum is attained.

I.e., we want to know under what conditions does it hold that φ has exactly $|P|$ lifts.

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This tells us that there exists a subgroup $U \leq G$ and a U-invariant character $\beta \in \mathrm{Irr}(N)$, where $N = \mathbf{O}_{p'}(U)$, such that induction from U to G defines bijections from the sets of ordinary and Brauer irreducible characters of U that lie over β onto the sets of ordinary and Brauer irreducible characters of the block B.

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Also, U can be chosen so that the defect group P of B is a Sylow p-subgroup of U.

In particular, there is a Brauer character $\theta \in \mathrm{IBr}(U)$ lying over β such that $\theta^G = \varphi$, and the lifts of θ in $\mathrm{Irr}(U)$ are in bijective correspondence with the lifts of φ in $\mathrm{Irr}(G)$.

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Conversely, if H is normal in U and P is abelian, then $H = N$ and α is U-invariant.

It is not difficult to see that the $|P|$ extensions of α to U will all be lifts of θ.

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Lemma

Let $H \leq G$ and $\chi \in \mathrm{Irr}(G)$, and write $N = \mathrm{core}_G(H)$. Then the following are equivalent.

1. $(\chi_H)^G$ *is a multiple of* χ.
2. χ *vanishes on* $G \setminus N$.
3. χ *vanishes on* $G \setminus H$.

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Thus, $V \leq H$ if and only if $V \leq N$. (This proves (2) and (3) are equivalent.)

For arbitrary $x \in G$, we have

$$(\chi_H)^G(x) = \frac{1}{|H|} \sum_{g \in G} \psi(gxg^{-1}),$$

Two general lemmas

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$$\psi(y) = \begin{cases} 0 \text{ if } y \notin H \\ \chi(y) \text{ if } y \in H. \end{cases}$$

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Since χ is a class function of G, we deduce that $\psi(y) = \chi(x)$ whenever $y \in H$ is conjugate in G to x.

This yields

$$(\chi_H)^G(x) = \frac{m}{|H|}\chi(x)\,,$$

where m is the number of elements $g \in G$ such that $gxg^{-1} \in H$.

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Suppose $x \in G$ satisfies $\chi(x) \neq 0$.
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We deduce that $x \in N$, and thus, χ vanishes on $G \setminus N$, as required.

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Since no conjugate of x lies in N, we have $m = 0$, and so, $(\chi_H)^G(x) = (m/|H|)\chi(x) = 0$.

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If $x \in N$, we have $m = |G|$, so $(\chi_H)^G(x) = |G : H|\chi(x)$.

We conclude that $(\chi_H)^G = |G : H|\chi$, and $(\chi_H)^G$ is a multiple of χ.

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2. α_N *is* G*-invariant and* α *vanishes on* $H \setminus N$.

The proof of this lemma has a similar flavor.

We have an observation.

Recall that $V(\alpha)$ is the subgroup of H generated by elements $h \in H$ so that $\alpha(h) \neq 0$.

Two general lemmas

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Recall that $V(\alpha)$ is the subgroup of H generated by elements $h \in H$ so that $\alpha(h) \neq 0$.

Since α vanishes on $H \setminus N$, we see that $V(\alpha) = V(\alpha_N)$.

Because α_N is G-invariant, this implies $V(\alpha)$ is normal in G.

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First, we recall a result of S. Dolfi which asserts that whenever a solvable group S acts faithfully on a group M, where $|S|$ and $|M|$ are coprime, there exist elements $x, y \in M$ such that $C_S(x) \cap C_S(y) = 1$.

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Assuming (as we may) that $|C_S(x)| \leq |C_S(y)|$, it is immediate that $|C_S(x)| \leq |S|^{1/2}$.

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We use a result of Hartley and Turull that says if a group S is acting coprimely on a group G where at least one of G or S is solvable, then A acts on an abelian group H where the sizes of the S-orbits on H are the same as the sizes of the S-orbits on G.

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Finally, we need a result of Itô that says that if $G = AB$ where A and B are abelian groups, then either A or B has a nontrivial core in G.

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Finally, we need a result of Itô that says that if $G = AB$ where A and B are abelian groups, then either A or B has a nontrivial core in G.

Using these ingredients, we prove the following:

Key Lemma

Lemma

Let S be solvable and nontrivial, and suppose that S acts faithfully and coprimely on a group M. Assume that $C_S(x)$ is abelian for each element $x \in M$ with the property that $|C_S(x)| = |S|^{1/2}$. Then there exists $m \in M$ such that $|C_S(m)| < |S|^{1/2}$.

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Idea of Proof:

By the result of Hartley and Turull, we may assume that M is abelian.

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Applying Itô's theorem, we may assume without loss of generality that there exists K normal in S with $1 < K \leq C_S(x)$.

Since M is abelian, Fitting's lemma applies to the action of K on M, and we have $M = C \times D$, where $C = C_M(K)$ and $D = [M, K]$.

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We conclude that $|C_S(xd)| < |C_S(x)| = |S|^{1/2}$, and this is a contradiction.

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To prove the theorems about induction and restriction, we also apply a result by Howlett and Isaacs that says if N is normal in G and N has an irreducible character that is fully ramified with respect to G/N (i.e., G/N is a central type factor group), then G/N is solvable. (The proof of this uses the Classification!)

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We will also need the following elementary well-known fact.

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We will also need the following elementary well-known fact.

Lemma

Let $S \leq G$ and M be normal in G with $S \cap M = 1$. Then $C_S(m) = S \cap S^m$ for each element $m \in M$.

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Thus $[s, m] \in S \cap M = 1$.

It follows that $s \in C_S(m)$, and thus $x = s^m \in C_S(m)$, as required.

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Idea of Proof:

It suffices to show that $\beta^G = \chi$ for one such constituent β of χ_N.

This is trivial if $N = G$, so we can assume that $N < G$, and we proceed by induction on $|G : N|$.

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Let ν be an arbitrary irreducible constituent of β^X.

It follows by the Clifford correspondence that ν^{H^m} is irreducible, and we write $\mu = \nu^{H^m}$.

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We show that $|X : N| \geq |S : N|^{1/2}$, and if equality holds here, X/N is abelian.

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Writing $\overline{G} = G/N$ and using the standard bar convention, we see that $\overline{S}$ acts faithfully on $\overline{M}$.

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We have now seen that for an arbitrary element $\overline{m}$ of $\overline{M}$, the stabilizer $\overline{X}$ of $\overline{m}$ in $\overline{S}$ satisfies $|\overline{X}| \geq |\overline{S}|^{1/2}$, and that if equality holds, then $\overline{X}$ is abelian.

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We deduce that $\overline{S} = 1$, or equivalently, that $S = N$.

Since S is the full stabilizer of β in G, it follows that β^G is irreducible, and thus $\beta^G = \chi$, as wanted.