Lecture 1

Math 50051, Topics in Probability Theory and Stochastic Processes

Asset prices, the object of your study this year, fluctuate randomly. A mathematical model of randomness is needed. Throughout the semester we will study various elementary models for asset prices, models that are well suited for pricing derivative assets.

Many investors seem to be driven by intuitive notions of probabilities rather than the axiomatic, formal, probability models. Nevertheless, it turns out, that if there are no arbitrage opportunities, one can represent the fair market value of financial assets using probability measures constructed synthetically. Therefore mathematical probability models have a natural use in pricing derivative products.

We need, first, to lay out the framework where the notion of chance and the resulting probability can be defined without falling into some inconsistencies. Therefore we need a set of basic states of the world. What do we mean by this?

Before we perform an experiment we can not say what the outcome is, but we can describe **all** possible outcomes. The set of all possible outcomes, or all possible states of the world, is denoted by Ω (some books use the notation of S).

Examples:

1. Toss a coin. In this case $\Omega = \{H, T\}$.
2. Roll a die. In this case $\Omega = \{1, 2, 3, 4, 5, 6\}$
3. Ask someone to pick a number between 1 and 2. In this case $\Omega = (1, 2)$.
4. Suppose the price of an exchange-traded commodity future during a given day depends only on a harvest report the USDA will make public during that day. In this case Ω is the set of all possible reports that the USDA may make public .

A particular outcome in Ω, or a particular state of the world, is typically denoted by ω. Often we refer to it as a realization.

Examples:

1. Toss a coin. In this case $\omega_1 = \{H\}$,$\omega_2 = \{T\}$.
2. Roll a die. In this case $\omega = \{1\}, \omega_2 = \{2\}$, etc.
3. Ask someone to pick a number between 1 and 2. In this case $\omega = 1.3$, etc.
4. Suppose the price of an exchange-traded commodity future during a given day depends only on a harvest report the USDA will make public during that day. In this case a possible ω is $\omega =$the production of wheat decreased by 1%.

5. Toss a coin. Repeat the experiment 10 times. $\Omega = \{H, T\}$. Outcomes: $\omega_1, \omega_1, \omega_2, \omega_2, \omega_2, \omega_2, \omega_1, \omega_2, \omega_1, \omega_1$. The cardinality of Ω (the number of elements) is 2.But if
6. Toss 10 coins in the air. Then $\Omega = \{(H, H, \cdots, H); (H, H, \cdots, H, T), \cdots, (T, T, \cdots, T)\}$, with $\omega_1 = (H, H, \cdots, H)$, $\omega_2 = (H, H, \cdots, H, T)$. The cardinality of ω is 2^{10}.
7. $\Omega =$ corral of cows. ω is a cow in the corral.

Let's look at another experiment. Toss 4 coins. I want to look at the event: exactly 3 heads occur. I can ask myself "What is the likelihood of this event?", or $P(event) =?$. In this case $\Omega = \{(H, H, H, H), (H, H, H, T), (H, H, T, H), \cdots, (T, T, T, T)\}$.

We can also look at Ω being a corral of cows, 16 of them, first with 4 heads, last with 4 tails. Which cows match the description of having 3 heads? This event, A, is composed of:

$$A = \{(H, H, H, T), (H, H, T, H), (H, T, H, H), (T, H, H, H)\}$$

Question: What is the likelyhood of a cow having 3 heads? In other words $P(A) =?$ To answer this we might need to know if each cow is equally probable. If they are not equally probable, corral does not change, cow space does not change, the event does not change, but the likelihood, the probability changes. If $P(H) = \frac{1}{3}$, $P(T) = \frac{2}{3}$ (think of this as having someone in another room roll a fair die: if getting 1 or 2 then "heads" is yelled back, if getting 3, 4, 5, or 6 then "tail"is yelled back. Then $P(A) = 4\frac{2^1}{3^4}$.

So it seems that, for each random event, we need to look at a triple: $(\Omega, eventspace, P)$. The set, collection, of all events is denoted by $\mathcal{F}$, and the triple $(\Omega, \mathcal{F}, P)$ is called the probability space.

So, an event is a subset of Ω, ie is a set formed with several ωs. To events we can assign probabilities. They are usually denoted by capital letters A, B, E, etc.

Examples:

1. Toss 4 coins. I want the event: exactly 3 heads occur.

$$A = \{(H, H, H, T), (H, H, T, H), (H, T, H, H), (T, H, H, H)\}$$

2. Toss 4 coins. I want the event: exactly 2 heads occur.

$$A = \{(H, H, T, T), (H, T, T, H), (H, T, H, T), (T, H, H, T), (T, T, H, H), (T, H, T, H)\}$$

3. Roll a die and let E be the event that we get an even number. Then $E = \{2, 4, 6\}$.
4. In the case of USDA report, depending on what is in the report, we can call it either favorable or unfavorable. Note that there might be several ωs that may lead us to call the harvest report "favorable". It is in this sense that events are collections of ωs

The collection of events we are interested in, $\mathcal{F}$, for which we will assign probabilities, might be smaller than all possible events. For example we might look at all USDA reports starting in 2000. This collection is smaller than the collection of all reports from all years. The space of events to which we assign probabilities, $\mathcal{F}$, is called in measure theoretical term a σ-field over Ω. It must obey the following conditions:

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$ (A is an event) then $A^c \in \mathcal{F}$.
3. If $A_1, A_2, \cdots$ is a countable sequence of events then their union must be an event, i.e. $\cup_i A_i \in \mathcal{F}$.

Examples:

1. The largest $\sigma-$field is the set of all subsets of Ω.
2. The smallest $\sigma-$field is $\mathcal{F} = \{\emptyset, \Omega\}$
3. Any Ω with only two points can have only the above as $\sigma-$fiels.
4. $\Omega = \{(H,T),(H,H),(T,H),(T,T)\}$
$$\mathcal{F}_1 = \{\emptyset, \Omega, \{(H,T),(H,H)\}, \{(T,H),(T,T)\}\}$$
$$\mathcal{F}_2 = \{\emptyset, \Omega, \{(H,H)\}, \{(H,T),(T,H),(T,T)\}\}$$
Observe that $B = \{(T,T)\} \notin \mathcal{F}_2$. Is it in $\mathcal{F}_1$?
5. $\mathcal{F} = \mathcal{B}(R) =$ the family of Borel sets, is the smallest $\sigma-$field containing all intervals in R.