Lecture 10

Math 50051, Topics in Probability Theory and Stochastic Processes

Martingales

The concept of martingale has its origin in gambling. It describes a fair game of chance. Favorable and unfavorable games are described by submartingales and supermartingales. Because option pricing is based on a "fair game" assumption, martingales play an important, crucial role in it.

All martingale properties that we will see are stated in discrete time for ease of proofs, but could be extended to continuous timed.

Discrete time

A sequence $X_1, X_2, ...$ of r.v. is called a martingale w.r.t. the filtration $\mathcal{F}_1, \mathcal{F}_2, ...$ if
1) X_n is integrable for each n;
2) X is adapted to $\mathcal{F}$, ie X_i is $\mathcal{F}_i$-measurable, or determined by $\mathcal{F}_i$ for each i;
3) $E[X_{n+1}|\mathcal{F}_n] = X_n$ a.s. for each n.

Continuous time

A family $(X_t)_{t\in T}$ of r.v. is called a martingale if
1) X_t is integrable for each $t \in T$;
2) X_t is $\mathcal{F}_t$-measurable;
3) $X_s = E[X_t|\mathcal{F}_s]$ for every $s \leq t$.

If the filtration used is the one generated by the r.v. itself, i.e. $\mathcal{F}_t^X = \sigma(X_s, s \leq t)$, then $(X_t)_{t\in T}$ is a martingale if
1) X_t is integrable for each $t \in T$;
2) $E[X_t|X_r, 0 \leq r \leq s] = X_s$ (or $E[X_{n+1}|X_1, X_2, ..., X_n] = X_n$ in the discrete case).

In other words a martingale is a process such that the best guess of tomorrow's outcome based on the information I have up to today is today's outcome.

According to this definition, martingales are stochastic processes whose future variations are completely unpredictable given the current information set. For example if S_t is a martingale with respect to the set of information $\mathcal{F}_t$ and we are looking at the forecast of the change in S_t over a interval of length $u > 0$ then:

$$E(S_{t+u} - S_t|\mathcal{F}_t) = E(S_{t+u}|\mathcal{F}_t) - E(S_t|\mathcal{F}_t) = S_t - S_t = 0$$

Hence the best forecast of the change in S_t over an arbitrary interval u is zero, in other words, the directions of the future movements in martingales are impossible to forecast. This is the fundamental characteristic of processes that behave like martingales. If the trajectories of a process display clear long or short term trends then they are not martingales.

CAREFUL: A martingale is always define with respect to some information sets, and with respect to some probability measure. If we change the information sets or the probability associated to the process, the process might not be a martingale anymore. The opposite is also true, if a process is not a martingale, by changing the information sets or the probability measure we can transform it into a martingale.

Example

I) Let $\mathcal{F}_1, \mathcal{F}_2, ...$ be a filtration and X be an integrable r.v. If $X_n = E[X|\mathcal{F}_n]$ then (X_n),$n = 1, 2, ...$ is a martingale w.r.t $\mathcal{F}_n$. This is an important example used frequently in pricing complicated interest rate derivatives, for discrete time intervals. Here X is a random variable whose values we do not know, but it will be revealed to us at a future date. X_n are the "forecasts" of the same rv X, made at different times. This sequence of forecasts is a martingale.

We know that the stock prices or the bond prices are not completely unpredictable. The prices of a discount bond is expected to increase over time. In general, the same is true for the stock prices, they are expected to increase in average. Hence, if B_t represents the price of a discount bond maturing at time T, $t < T$,

$$B_t < E(B_u|\mathcal{F}_t) \;\; t < u < T$$

Therefore of interest are the following type of processes

$(X_t)_{t\in[0,\infty)}$ is a supermartingale (submartingale) with respect to the filtration $\mathcal{F}_t$ if:
1) X_t is integrable for all t;
2) X_t is adapted to $\mathcal{F}_t$;
3) $E[X_t|\mathcal{F}_s] \le X_s$ ($E[X_t|\mathcal{F}_s] \ge X_s$) for all $s \le t$ and for all $t \in [0, \infty)$.

Remark: If asset prices are more likely to be sub- or supermartingales, then why such interest in martingales in Financial Mathematics? It turns out that although most financial assets are not martingales, one can convert them into martingales. For example, one can find an artificial probability distribution such that bond or stock prices discounted by the risk-free rate of return become martingales!

We also said that a martingale is the equivalent of a fair game while a supermartingale (submartingale) is the equivalent to an unfavorable game (favorable game).
Let's see how.

Gambling interpretation

We will do the discrete time case and the modification to continuous-time interpretation is obvious.

Let $X = \{X_n, n = 0, 1, 2, ...\}$ be the process such that $X_n - X_{n-1}$ is your net winning per unit stake in game n in a series of games played at time $n = 1, 2, ...$ For the filtration we use the usual one $\mathcal{F}_n^X$, i.e. if $n-1$ rounds of the game have been played so far, your accumulated knowledge will be represented by $\mathcal{F}_{n-1}^X$. Also observe that X_n is the total winnings after n games.

The game is fair if $E[X_n|\mathcal{F}_{n-1}^X] = X_{n-1}$, i.e. you expect that your fortune at step n will be in average the same as at step $n - 1$.
Indeed $E[X_n|\mathcal{F}_{n-1}^X] = X_{n-1} = E[X_{n-1}|\mathcal{F}_{n-1}^X] \Rightarrow E[X_n - X_{n-1}|\mathcal{F}_{n-1}^X] = 0$.
The game series is unfavorable if

$$E[X_n - X_{n-1}|\mathcal{F}_{n-1}^X] \le 0$$

$$i.e. \;\; E[X_n|\mathcal{F}_{n-1}^X] \le E[X_{n-1}\mathcal{F}_{n-1}^X] = X_{n-1}.$$

and it is favorable if

$$E[X_n - X_{n-1}|\mathcal{F}^X_{n-1}] \geq 0$$
$$i.e.\ \ E[X_n|\mathcal{F}^X_{n-1}] \geq X_{n-1}.$$

Suppose that you can vary the stake to be S_n in game n, i.e. $S = \{S_n; n = 1, 2, ...\}$ is a stochastic process such that $S_n(X_n - X_{n-1})$ is your total winning on game n. But when the time comes to decide your stake S_n, you will know the outcomes of the first $n-1$ games, hence it is reasonable to assume that S_n is $\mathcal{F}^X_{n-1}$-measurable. Such a stochastic process is called previsible.

Let's continue by looking at the total winning up to time n, they are

$$T_n = \sum_{k=1}^{n} S_k(X_k - X_{k-1}) = S_1(X_1 - X_0) + S_2(X_2 - X_1) + ... + S_n(X_n - X_{n-1}) = \text{ by def } (S \cdot X)_n$$

and $(S \cdot X)_0 = 0$
$(S \cdot X)_n$ is called the martingale transform of X by S. It is the discrete analogue of the stochastic integral $\int_0^t S_s dX_s$, which is one of the greatest achievements of modern probability theory and which we will study at the end of the semester.

The following proposition shows that you can not beat the system! A fair game will always be a fair game and an unfavorable (favorable) game can not be transformed into a favorable (resp. unfavorable) game, if one can not be allowed to wager negative sums of money.

Proposition
Let $S_1, S_2, ...$ be a gambling strategy.
1) If $S_1, S_2, ...$ is a bounded sequence (i.e. one has limited resources), and $X_0, X_1, X_2, ...$ is a martingale, then $T_n = \sum_{k=1}^{n} S_k(X_k - X_{k-1})$ is a martingale.
2) If $S_1, S_2, ...$ is a non-negative bounded sequence(i.e. one has limited resources and is allowed to wager only non-negative amounts of money), and $X_0, X_1, X_2, ...$ is a supermartingale (submartingale), then $T_0, T_1, T_2, ...$ defined as above is a supermartingale (submartingale) .

Example 2
A wealthy gambler follows the following strategy in waging on fair bets: he starts waging \$1 on bet 1, if he loses he wages \$2 on set 2,... If he loses the first n bets he wages \$$2^n$ on the $(n+1)$st bet. He is bound to win sooner or later, say on Tth bet, at which point he ceases to play and leaves with his profit of

$$2^T - (1 + 2 + 4 + ... + 2^{T-1})$$

This sounds like on enticing strategy of always wining, but be careful!

Example 3
Let $X_1, X_2, ...$ be a sequence of independent integrable r.v. such that $E[X_n] = 0$ and let $S_n = X_1 + X_2 + ... + X_n$, $\mathcal{F}_n = \sigma(X_1, X_2, ...X_n)$, then $(S_n, \mathcal{F}_n)$ is a martingale.

Show that if X_n is a martingale w.r.t. $\mathcal{F}_n$ then $E(X_1) = E(X_2) = ...$

Stopping times

Let's consider again the gambling interpretation of a martingale.
The gambler's fortune after the nth play is X_n and the information about the game at that time is represented by the σ-field $\mathcal{F}_n$. If the filtration $\mathcal{F}_n$ is the usual filtration $(\mathcal{F}_n^X)$, $\mathcal{F}_n^X = \sigma(X_1, ...X_n)$. Hence he knows the sequence of his fortunes and nothing else, but in general $\mathcal{F}_n$ could be larger. The martingale condition stipulates that his expected or average fortune after the next play equals his present fortune, and so the martingale is a model for a fair game.

Suppose, now, that the gambler decides to leave, instead of playing indefinitely, either because he thinks that he has won (or lose) enough, or because he is discouraged by the way the game has been going, or for any other reasons. Then the game is still fair (or advantageous) if it was so originally.(Unless the gambler has quit because he can foresee the future, or has some inside information, and knows, for example, that the next plays will go against him, but such "unfair" stopping rules do not concern us now.)

The number of rounds played before quitting the game will be denoted by δ, and its mathematical model is given by a random variable that could take values $\{1, 2, ..., n, ..., \underline{\infty}\}$ (∞ is included to cover the theoretical possibility that the game never stops). δ is called a stopping rule or a stopping time if at each step n one is able to decide whether to stop playing or not, i.e. whether or not $\delta = n$. Therefore the event $\{\delta = n\}$ must be in $\mathcal{F}_n$ if we want δ to be a valid stopping rule. An honest gambler can not peer into the future.

Example
1) An honest gambler could decide at the beginning of the game to stop after, say 10 rounds.So it must be the case that $\delta = k$, where k is a fixed, constant, number of rounds, is a stopping time.

2) Let X_n be a sequence of r.v. adapted to the filtration $\mathcal{F}_n$, and let $B \subset \mathbb{R}$ be a Borel set (an interval for example). Show that the time of first entry of X_n into B, i.e. $\delta = min\{n, X_n \in B\}$ is a stopping time.

If δ is the time the gambler stops, and it is a stopping rule, his fortune at time n for this stopping rule is

$$X_n^\delta = \begin{cases} X_n & if\ n \leq \delta \\ X_\delta & if\ n \geq \delta \end{cases}$$

Here X_δ (Which has value $X_{\delta(w)}(w)$ at w) is the gambler's ultimate fortune, and it is his fortune for all times subsequent to δ.
The random variable X_n^δ is called the stopped r.v.X_n.

Notation
If $a \wedge b$ denotes the minimum between 2 numbers a and b, then

$$X_n^\delta = X_{\delta\wedge n},\ \ i.e.\ \ X_n^\delta(w) = X_{\delta\wedge n}(w) = \begin{cases} X_n(w) & if\ n \leq \delta(w) \\ X_{\delta(w)}(w) & if\ n \geq \delta(w) \end{cases}$$

Proposition
A fair stopping rule does not change the fairness of the game, i.e. if δ is a stopping time and $(X_n)_{n\in\mathbb{N}}$ is a martingale, then so is $(X_{\delta\wedge n})_{n\in\mathbb{N}}$. Similarly if X_n is a submartingale (supermartingale), then so is X_n^δ.

Optional sampling theorem is one of the most important properties of the martingales. It states in effect that "You can not beat a fair game", i.e. if X is a martingale and δ is a stopping time, then $E(X_\delta) = E[X_0]$. However, it is easy to see that this theorem is false in complete generality!! Indeed, if we are looking at the previous betting strategy that was a sure way of making money, and let δ be the first time that the coin comes up heads, we saw that at that time X_δ – the total amount he gain / loss was \$1, hence $E(X_\delta) = 1$ but $E(X_0) = 0$!!

Optional Stopping theorem

Let X_n be a martingale and δ a stopping time with respect to a filtration $\mathcal{F}_n$ such that the following condition hold:

1) $P(\delta < \infty) = 1$;

2) X_δ is integrable;

3) $E(X_n|_{\{\delta > n\}}) \to 0$ as $n \to \infty$, then $E(X_\delta) = E(X_1)$.