Lecture 11

Math 50051, Topics in Probability Theory and Stochastic Processes

Stopping times

Remember, we said that if τ is the time the gambler stops, and it is a stopping rule, his fortune at time n for this stopping rule is

$$X_n^\tau = \begin{cases} X_n & if\ n \le \tau \\ X_\tau & if\ n \ge \tau \end{cases}$$

Here X_τ (Which has value $X_{\tau(w)}(w)$ at w) is the gambler's ultimate fortune, and it is his fortune for all times subsequent to τ.
The random variable X_n^τ is called the stopped r.v.X_n.

Notation

If $a \wedge b$ denotes the minimum between 2 numbers a and b, then

$$X_n^\tau = X_{\tau\wedge n}, \;\; i.e.\;\; X_n^\tau(w) = X_{\tau\wedge n}(w) = \begin{cases} X_n(w) & if\ n \le \tau(w) \\ X_{\tau(w)}(w) & if\ n \ge \tau(w) \end{cases}$$

Proposition

A fair stopping rule does not change the fairness of the game, i.e. if τ is a stopping time and $(X_n)_{n\in\mathbb{N}}$ is a martingale, then so is $(X_{\tau\wedge n})_{n\in\mathbb{N}}$. Similarly if X_n is a submartingale (supermartingale), then so is X_n^τ.

Optional sampling theorem is one of the most important properties of the martingales. It states in effect that "You can not beat a fair game", i.e. if X is a martingale and τ is a stopping time, then $E(X_\tau) = E[X_0]$. However, it is easy to see that this theorem is false in complete generality!! Indeed, if we are looking at the previous betting strategy that was a sure way of making money, and let τ be the first time that the coin comes up heads, we saw that at that time X_τ – the total amount he gain / loss was \$1, hence $E(X_\tau) = 1$ but $E(X_0) = 0$!!

Optional Stopping theorem

Let X_n be a martingale and τ a stopping time with respect to a filtration $\mathcal{F}_n$ such that the following condition hold:
1) $P(\tau < \infty) = 1$;
2) X_τ is integrable;
3) $E(X_n 1_{\{\tau > n\}}) \to 0$ as $n \to \infty$, then $E(X_\tau) = E(X_1)$.

Wald's equation, Let $S_n = \sum_{r=1}^n X_r$ be a random walk started at the origin (i.e. X_r are iid r.v. that take only values 1 or -1), such that $E(X_r) = \mu < \infty$. Let T be a stopping time for S_n, with $E(T) < \infty$. Then $E(S_T) = \mu E(T)$.

In addition to stopping nicely, martingales verify some powerful inequalities. Here are two of them:'
Doob's maximal inequality Suppose that X_n is a non-negative submartingale with respect with a filtration. Define $U_n = \max_{k\le n} X_k$. Then

$$P(U_n \ge \lambda) \le \frac{1}{\lambda} E(X_n 1_{\{U_n \ge \lambda\}}).$$

Similarly, if $U = \max_{n\geq 0} X_n$ then

$$P(U \geq x) \leq E(X_0)/x$$

.

Doob's maximal L^2 inequality

If X_n, $n \in \mathbb{N}$ is a non-negative square integrable submartingale with respect to $\mathcal{F}_n$, then

$$E|max_{k\leq n} X_k|^2 \leq 4E|X_n|^2$$

Example: Suppose the non-negative martingale X_n is indeed the sequence of values of a gambler's wealth playing only fair games, with no credit, where the stake is always less than the existing fortune X_n. Without loss of generality, let $X_0 = 1$. Then the maximal inequality shows that the probability that this wealth ever reaches the level $x \geq 1$ is less than $\frac{1}{x}$. For example, we have that no matter what fair game you play, for whatever stake, with no credit, the chance of ever doubling your money is less than $\frac{1}{2}$.

The upcrossings inequality

Let X_n be a supermartingale and let's look at the following gambling strategy, or investing strategy: we do not gamble (invest) until X_n becomes less than a preestablished number A. As soon as this happens we start gambling (investing) until stakes at each round of the game and continue until X_n becomes greater than a preestablished number B. At this point we refrain again from playing until X_n becomes smaller than A and so on. The strategy S_n is such that $S_n = 0$ if we do not play the nth game and $S_n = 1$ otherwise. So, if I look to see what happens as long as $S_n = 1$, I see that my process X_n crossed the interval $[A, B]$, starting bellow A and finishing above B. For convenience we identify each upcrossing with its last step k when $S_k = 1$ and $S_{k+1} = 0$. See picture:

$$PICTURE$$

In math formula this strategy can be written:

$$S_{n+1} = \begin{cases} 1 & if\ S_n = 0\ and\ X_n > a \\ 1 & if\ S_n = 1\ and\ X_n \leq b \\ 0 & otherwise \end{cases}$$

and it is called upcrossing strategy. The upcrossing forms an increasing sequence: $u_1 < u_2 < ...$ and we denote by $U_n[A, B]$ the number of upcrossings up to time n.

Observe that for each upcrossing our winnings will be increased by a total of at least $B - A$!

Theorem

If $X_1, X_2, ...$ is a supermartingale and $A \leq B$, then

$$(B - A)E(U_n[A, B]) \leq E((X_n - A)^-)$$

where $(X_n - A)^- = \begin{cases} A - X_n & when\ A \geq X_n \\ 0 & when\ A < X_n \end{cases}$

Doob's Martingale convergence Theorem
Suppose that $X_1, X_2, ...$ is a supermartingale with respect to the filtration $\mathcal{F}_n$ such that

$$sup_n E(|X_n|) < \infty$$

or

$$E(X_n^2) \leq K < \infty, \quad \text{for all } n$$

Then these is an integrable r.v. X such that

$$lim_{n\to\infty} X_n = X \;\; a.s.$$

Remark
This theorem is true if X_n is a martingale. This is because all martingales are supermartingales. Also because if X_n is a submartingale, $-X_n$ is a supermartingale, the theorem will be true for submartingales too.

Uniform integrability
1) We saw in the previous theorem that in certain cases, martingales are convergent, and the a.s. limit is an integrable r.v. If we want convergence of the expectations of such martingales, I need extra properties for my martingale.
2) A second reason these properties (uniform integrability) are nice to have is because sometimes they are much easier to verify than the 3 conditions from the optional sampling theorem, but they imply those 3 conditions, so it is enough to verify uniform integrability (u.i.) for OST to be true.

Definition
Let $X_1, X_2, ...$ be a sequence of r.v. This sequence is called integrable if for every $\epsilon > 0$ there exists an $M > 0$ such that

$$(*) \qquad \int_{\{|X_n|>M\}} |X_n| dp < \epsilon$$

for all $n = 1, 2, ...$

Remark
1) Inequality (*) is equivalent to

$$E(|X_n| 1_{\{|X_n|>M\}}) < \epsilon$$

Now, this is easy to see that (*) is equivalent to

$$\int_M^\infty |x| f_n(x) dx < \epsilon$$

where $f_n(x)$ is the density of X_n, if it exists.
2) The u.i. conditions are very strong conditions. It is very easy to have a sequence of r.v. that it is not u.i.
3) M does not depend on n !

Example
Consider the martingale betting strategy (the strategy that makes money for you for sure).

Another reason u.i. is important is the following theorem:

Theorem Every uniform integrable martingale X_n, with respect to $\mathcal{F} - n = \sigma(X_1, \cdots, X_n)$ is convergent in L^1 and if we denote its limit in L^1 by X then

$$X_n = E(X|\mathcal{F}_n)$$

By convergent in L^1 we mean that there is an X such that $E(|X_n - X|) \to 0$.

Doob-Meyer Decomposition

Suppose a trader observes the price of a financial asset S_t at times t_i

$$t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k = T$$

If the intervals between the times t_{i-1} and t_i are very small, and if the market is liquid, the price of the asset is likely to exhibit at most one uptick or one downtick during a tipical interval t_{i-1} to t_i. We formalize this by saying that at each instant t_i there are only two possibilities for S_{t_i} to change:

$$\Delta S_{t_i} = 1 \quad \text{with probability } p$$

or

$$\Delta S_{t_i} = 0 \quad \text{with probability } 1 - p$$

It is assumed that these changes are independent of each other. Also observe that if $p = \frac{1}{2}$ than $E(\Delta S_{t_i}) = 0$, otherwise it is not zero.

We already looked at this example. Is S_{t_i} a martingale? What about $Z_{t_i} = S_{t_i} + (1 - 2p)(k + 1)$? What sort of process is S_{t_i}?

In general:

Theorem: If X_t is a right continuous submartingale with respect to the family $\mathcal{F}_t$ and if $E(X_t) < \infty$ for all t then X_t admits the decompostion

$$X_t = M_t + A_t$$

where M_t is a right continuous martingale with respect to the probability P and filtration $\mathcal{F}_t$ and A_t is an increasing process adapted to $\mathcal{F}_t$.

Homework: Please read Section 8.2.2 from Neftci for a possible use of Doob Decomposition Theorem in Finance.