Lecture 12

Math 50051, Topics in Probability Theory and Stochastic Processes

Doob-Meyer Decomposition

Suppose a trader observes the price of a financial asset S_t at times t_i

$$t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k = T$$

If the intervals between the times t_{i-1} and t_i are very small, and if the market is liquid, the price of the asset is likely to exhibit at most one uptick or one downtick during a tipical interval t_{i-1} to t_i. We formalize this by saying that at each instant t_i there are only two possibilities for S_{t_i} to change:

$$\Delta S_{t_i} = 1 \quad \text{with probability } p$$

or

$$\Delta S_{t_i} = -1 \quad \text{with probability } 1 - p$$

It is assumed that these changes are independent of each other. Also observe that if $p = \frac{1}{2}$ than $E(\Delta S_{t_i}) = 0$, otherwise it is not zero.

We already looked at this example. Is S_{t_i} a martingale? What about $Z_{t_i} = S_{t_i} + (1 - 2p)(k + 1)$? What sort of process is S_{t_i}?

In general:

Theorem: If X_t is a right continuous submartingale with respect to the family $\mathcal{F}_t$ and if $E(X_t) < \infty$ for all t then X_t admits the decompostion

$$X_t = M_t + A_t$$

where M_t is a right continuous martingale with respect to the probability P and filtration $\mathcal{F}_t$ and A_t is an increasing process adapted to $\mathcal{F}_t$.

Homework: Please read Section 8.2.2 from Neftci for a possible use of Doob Decomposition Theorem in Finance.

Brownian motion

Most markets for financial assets and derivative products may, from time to time, exhibit "extreme" behavior. These events are exactly when we have the greatest need for accurate pricing. What makes an event "extreme" or "rare"? Is turbulence in financial markets the same as rare events? What difference between normal events and rare events is the way their size and their probability of occurrence changes (or does not change) with the observation interval. As the interval of observation Δt gets smaller the size of normal events gets smaller. This is what makes them ordinary. In one month several large price changes might be observed. In a week, fewer. Observing a number of large price jumps during a period of a few minutes is even less likely. Often the events that occur during an ordinary minute are not given much attention. But because they are ordinary even in a very small time interval, there is always a nonzero probability that some nonnoticible news will arrive. A rare event is different. By definition, it is suppose to occur infrequently. In continuous time, this means that as $\Delta t \to 0$ its probability of occurrence goes to 0. BUT its size might not shrink. A market crash such as the one this year is rare. On a given day, during a very short period, there is negligible probability that one will observe such a crash. But when it occurs its size is not different if one looks at an interval of 5 min or a whole trading day.

There are two basic building blocks in modeling continuous time asset prices. One is the Wiener process, or Brownian motion. This is a continuous stochastic process and can be used if markets are dominated by ordinary events while extremes occur only infrequently, according to the probabilities in the tail areas of the normal distribution. The second is the Poisson process which can be used for modeling systematic jumps caused by rare events. The Poisson process is discontinuous. By combining these two building blocks, one can generate a model that is suitable for a particular application.

Brownian motion(or Wiener process)

The name (Brownian) comes from the botanist R.Brown that observed the erratic behavior of a pollen particle in water.
We will study mainly the one-dimensional B.m. that could be seen as the projection of the position as the pollen particle onto one of the axes of the coordinate system. B.m. is a stochastic process that models random continuous motion. In order to model these motions we start by writing down the physical assumptions that we will make.
(1) Let W_t represent the position of a particle at time t. In this case t takes values on the nonnegative real numbers while W_t takes values on the real line (or perhaps the plane of space). Hence W_t is a continuous time-continuous space s.p.
(2) $W_0 = 0$ (for ease)
(3) The motion is completely random. We do not mean W_s and W_t are independent but rather the motion after time s, $W_t - W_s$ is independent of W_s.
(4)The distribution of the random movement should not change with time. Hence the distribution of $W_t - W_s$ depends only on $t - s$.
(5)For the time being we assume there is no drift (no general direction), i.e. $E(W_t) = 0$.
These assumptions are not enough to describe W_t because, if Y_t is a Poisson process then $X_t = Y_t - t$ satisfies them. Hence we need
(6) The function W_t is a continuous function of t.

Remarks:
a) The above assumption uniquely describe the process at least up to a scaling constant.
b) The Wiener process could be (and it is) widely used to describe models in valuing various noisy random systems. In particular, it is used to model asset prices.
c) If the noise in the system is determined by various independent sources then the CLT(central limit theorem) predicts that the net result will have a normal distribution. Indeed, our process verifies that.

Definition
A Brownian motion or a Wiener process is a stochastic process W_t with values in $\mathbb{R}$, defined for $t \in [0, \infty)$, s.t.

i) $W(0)=0$ a.s.
ii) For any $s_1 \leq t_1 \leq s_2 \leq t_2 \leq ... \leq s_n \leq t_n$ the r.v. $W_{t_1} - W_{s_1}, ..., W_{t_n} - W_{s_n}$ are independent (increments over nonoverlapping intervals are independent).
iii) For any $s < t$, the r.v. $W_t - W_s$ is distributed $N(0, t-s)$.
iv) The path are continuous, i.e. $t \to W_t$ is a continuous function of t.

Remarks:
1) B.m. is a s.p. with independent increments and stationary increments.
2) If W_t is a B.m. (starting at 0) then $Y_t = W_t + x$ is viewed as a B.m. starting at x.

Properties:
I) W_t is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(W_r, 0 \leq r \leq t)$.
II) $|W_t|^2 - t$ is a martingale with respect to the above filtration.
III) $V_t = \frac{1}{C} W(C^2 t)$ is a B.m. if $W(t)$ is.
IV) $V_t = W(t+T) - W(T)$ is a Wiener process if $W(t)$ is.

Levy's martingale characterization
Let $W(t)$,$t \geq 0$ be a s.p. and $\mathcal{F} = \sigma(W_s, s \leq t)$. Then $W(t)$ is a B.m. iff:
1) $W(0) = 0$ a.s.
2) $t \to W(t)$ are continuous.
3) $W(t)$ is a martingale with respect to $\mathcal{F}_t$.
4) $|W(t)|^2 - t$ is a martingale w.r.t. $\mathcal{F}_t$.

Markov property
We saw that $E(W_t|\mathcal{F}_s) = E(W_s|\mathcal{F}_s) + E(W_t - W_s|\mathcal{F}_s) = W_s$,
similarly $E(W_t|W_s) = E(W_s|W_s) + E(W_t - W_s|W_s) = W_s$.
Hence $\underline{E(W_t|\mathcal{F}_s) = E(W_t|W_s)}$
This illustrates the Markov property of B.m.,i.e. in order to predict W_t given all information up through time s, it suffices to consider only the value of the B.m.at time s.
Let $p_t(x,y)$ (or $p(t,x,y)$) denote the transition densities, i.e., the density of W_t for B.m. starting at x. Since $W_t - W_0$ is normal, mean 0, variance t

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}, \quad -\infty < y < \infty$$

(W_t is normally distributed with mean x and variance t) Then the Chapman-Kolmogorov equation is

$$p_{s+t}(x,y) = \int_{-\infty}^{\infty} p_s(x,y) p_t(z,y) dz$$

Remark
I) The above property states that $Y_t = W_{s+t} - W_s$ is a B.m. independent of W_s. This is equivalent to saying that $z_t = W_{s+t}$ is a B.m. starting at the random starting point W_s. The Chapman-Kolmogorov equation just averages the density $p_t(z,y)$ (or $p(t,z,y)$) over all possible starting points z.
II) in order to do many useful computations about B.m. a more general Markov property is needed. This is referred to as strong Markov property
Let T be a stopping time taking values in $[0,\infty)$ (i.e. the event $\{T \leq t\} \in \mathcal{F}_t$: knowing if the

process has stopped before time t one only needs to look at the information up to time t.) and let $\mathcal{F}_T$ be the information contained in the B.m. up to the stopping time T (one gets to view the path up through time T but not beyond) , Then if we denote

$$Y_t = W_{t+T} - W_T$$

the strong Markov property states that Y_t is a B.m. independent of $\mathcal{F}_T$.

Remarks on the Fractal Nature of B.m.

Remark 1 After we decide that Bm exhibits Markov property, the first question that one asks himself is if X_t is recurrent. In other words are there arbitrarily large times t with $X_t = 0$? To answer this question we need to look at the set

$$Z = \{t : X_t = 0\}$$

and observe its properties. It turns out this set is a “fractal” subset of the real line.

Remark 2 Using the “Reflection Principle” on Bm we can show that eventually the B.m. returns to the origin. Once returned to the origin, because of the strong Markov property, it will forget what it did and will start like a new B.m.. Applying the same reasoning as above this new B.m. will return, eventually, to the origin. We can continue the cycle and conclude that the B.m. returns to the origin infinitely often. in particular, B.m. is recurrent.

Remark 2 We can have the same discussion about any state (not only 0).

Remark 3 In any interval about 0 the B.m.takes both positive and negative values, again using the reflexion principle.

Doob's maximal L^2 inequality

$$E(max_{s\leq t}W^2(s)) \leq 4E(W(t)^2) = 4t$$

Brownian motion in several dimensions.

Let $X_t^1, X_t^2, ..., X_t^d$ be independent B.m.. The vector-valued stochastic process $X_t = (X_t^1, ..., X_t^d)$ is called d-dimensional B.m..

Properties:

The d-dimensional B.m. has the same properties from the definition of B.m. except that now the increment $X_t - X_s$ (a d-dimensional r.v.) has a joint normal distribution with mean 0 and covariance $(t-s)T$, i.e. has density

$$f(x_1, ..., x_d) = \frac{1}{\sqrt{2\pi r}} e^{-\frac{(x_1)^2}{2r}} \cdot ... \cdot \frac{1}{\sqrt{2\pi r}} e^{-\frac{(x_d)^2}{2r}} = \frac{1}{(2\pi r)^{d/2}} e^{-\frac{|x|^2}{2r}} \quad with \;\; r = t - s.$$

Remark

As in 1-dimensional case, the transition probability density of X_t assuming $X_0 = x$,is given by

$$p_t(x, y) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|y-x|^2}{2t}}$$

which satisfies the Chapman-Kolmogorov equation

$$p_{s+t}(x,y) = \int_{\mathbb{R}^d} p_s(x,z)p_t(z,y)dz_1,...,dz_d$$