Lecture 18

Math 50051, Topics in Probability Theory and Stochastic Processes

Sample path of B.m.

When we defined the Riemann integral, we defined it as a limit of Riemann sums, i.e. sums of rectangles with width a and subinterval of the integration range and length the value of the function at one of the points in the interval.

PICTURE

So, it looks like it is important, in defining the integral, to evaluate $f(t_i) - f(t_{i-1})$ where (t_{i-1}, t_i) is a small interval of my interval (a, b).

Let's $0 = t_0^n < t_1^n < ... < t_n^n = T$ be a partition of the interval $[0, T]$ into n equal parts. Hence

$$t_i^n = \frac{iT}{n} = i \cdot \frac{T}{n}$$

We denote by

$$\Delta_i^n W = W(t_{i+1}^n) - W(t_i^n)$$

the increment of the B.m. over the interval (t_i, t_{i+1})

PICTURE

Proposition

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} (\Delta_i^n W)^2 = T, \quad in \;\; L^2$$

Definition

The variation of a function $f : [0, T] \to \mathbb{R}$ is defined to be

$$limsup_{\Delta t \to 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|$$

where $t = (t_0, t_1, ..., t_n)$ is a partition of $[0, T]$ and where

$$\Delta t = max_{i=0,1,...,n-1} |t_{i+1} - t_i|$$

Theorem

The variation of the path of $W(t)$ is infinite a.s.

Remark:

What this means is that if we are to look at the details of B.m., it varies, wiggles, a lot, even in a short interval. Its variation is ∞.

This is important because we will not be able to define an integral with respect to B.m. in the same way as a Riemann integral, for each W (i.e. pathwise). We will need to have a probabilistic approach to it.

Brownian motion and stochastic integral

Let's look at a simple binomial model. Let S_n be the stock price at time $n \in \mathbb{N}$, be defined be:

$$S_0 = 1$$
$$S_{n+1} = S_n Z_n$$

where Z_n are iid r.v. defined by:

$$Z_n = \begin{cases} u & \textit{with prob. } P_u \\ d & \textit{with prob. } P_d \end{cases} \quad \text{with} \quad P_u + P_d = 1$$

And assume $E[Z_n] = \mu$, $Var[Z_n] = \sigma^2$. Let's write $Z_n = \sigma V_n + \mu$, $E[V_n] = 0$, $Var[V_n] = 1$.
Notice that $S_{n+1} - S_n = S_n Z_n - S_n = S_n(Z_n - 1) = S_n(\sigma V_n + \mu - 1)$,
that is, $S_{n+1} - S_n = \sigma S_n V_n + S_n(\mu - 1)$

More generally, let's look at a model that uses time instants $0, \Delta t, 2\Delta t, 3\Delta t, ..., n\Delta t, ...$ with Δt a small interval of time. Let's fix t to be one of the $n\Delta t$'s.

$$S(t + \Delta t) - S(t) = \sigma S(t) W(\Delta t \textit{ at time } t) + S(t)\Delta t(\mu - 1)$$

Let's write this in a more general form

$$(*)\ S(t + \Delta t) - S(t) = \sigma(S(t))\Delta W + \mu(S(t))\Delta t$$

where $\Delta W = W(\Delta t \textit{ at time } t)$ by notation.

Q: What is the magnitude of this process ΔW?
A: Choose ΔW to be a Gaussian process, distributed $N(0, \Delta t)$ (by a Gaussian process we mean a process whose rv are normally distributed and whose joint distributions are normal).

Therefore it is natural to choose $W(t)$ to be Bm and then ΔW will be $\Delta W(t) = W(t+\Delta t) - W(t)$.

Q: What happens to ΔW as $\Delta t \to 0$?

Proposition: $W(t)$ is a nowhere differentiable function of t.

I have the complete proof of this statement on the class web-page.

Therefore when I look at

$$\frac{S(t + \Delta t) - S(t)}{\Delta t} = \sigma S(t) \frac{\Delta W}{\Delta t} + \mu S(t),$$

and when $\Delta t \to 0$ we obtain something that does not make sense. Therefore, here we let Δt be small, infinitesimally small, denoted ṭ, but we would need to look at the equation

$$dS(t) = \sigma S(t)dW + \mu S(t)dt$$

This is called the **diffusion equation**. What is the precise mathematical meaning of this equation? Formally it resembles a differential equation, but as we saw above, we can not use the normal method of solving it because of the fact that Bm path is nowhere differentiable. A way around this was found by Ito in 1940s. In his theory of **stochastic integral** he gave a rigorous meaning to the equations such as above, by writing them as integral equations involving a new kind of integral:

$$S(t) - S(0) = \sigma \int_0^t S(s)dW(s) + \mu \int_0^t S(s)ds$$

where the integral with respect to $W(t)$ is called the **Ito integral**. Does it differ from a regular integral? Yes, by the way the "Riemann sums" converge.

Example: One of the first applications of the Wiener process was proposed by Bachelier, who around 1900 wrote a paper on modeling of asset prices at the Paris Stock Exchanges. He used $W(t)$ as a description of the market fluctuations affecting the price $X(t)$ of the asset. Namely, he assumed that infinitesimal price increments $dX(t)$ are proportional to the increment $dW(t)$ of the Wiener process: $dX(t) = \sigma dW(t)$ where σ is a positive constant. Therefore, an asset with initial price $X(0) = x$ would be worth

$$X(t) = x + \sigma W(t)$$

But this assumption has a serious flaw in the fact that implies that the stock prices are negative. So what is to do? To remedy the flaw it was observed that investors work in term of their potential gain or loss $dX(t)$ in proportion to the invested sum$X(t)$. Therefore, it is in fact the relative price of an asset that reacts to the market, hence should be proportional to $dW(t)$:

$$dX(t) = \sigma X(t)dW(t).$$

If this would be just an ODE and $\int_0^t X(t)dW(t)$ would be just a regular integral, the solution to this DE should be $xe^{W(t)}$. But, in fact, it turns out to be

$$X(t) = xe^{W(t)}e^{-\frac{t}{2}}.$$

The factor $e^{-\frac{t}{2}}$ is due to the non-differentiability of the path of the Wiener process. So, we see this does not work exactly like in calculus, or ODE. But how does it work?

Precise definition of the stochastic integral

We saw last time when we tried to generalize the binomial model that the only way we could make sense of equation (*) if we were to write it as

(1) $dS(t) = \mu(S(t))dt + \sigma(S(t))dW(t)$ with $W(t)$ being Brownian motion. In general, the function μ is called the drift while the function σ is called the diffusion.

It is natural to interpret (1) as:

$$S(t) = S(0) + \int_0^t \mu(S(s))ds + \int_0^t \sigma(S(s))dW(s)$$

(*) The first integral is just a Riemann integral.
The second integral is what we call a "stochastic integral". In fact this looks very much like a Riemann sum (Riemann-Stieltjes integral) only that the convergence type is different. What do we mean by this ?

Definition

A stochastic process $\{g(s) : s \geq 0\}$ is said to belong to the class $\mathcal{L}^2[a,b]$, and is said to be Ito-square integrable w.r.t.W in $[a,b]$, if:

* g is adapted to $\mathcal{F}^W$ $(g(t) \in F_t^W$ for all $t)$

* $\int_a^b E[g(s)^2]ds < \infty$

Proposition

If $g \in \mathcal{L}^2[0,t]$, then

$$I_n := \sum_{k=0}^{n-1} g(\frac{kt}{n})[W(\frac{(k+1)t}{n}) - W(\frac{kt}{n})] \longrightarrow \ \ \textit{at a random variable } I$$

as $n \to \infty$, $\mathcal{L}^2(\Omega)$. i.e. $E[(I_n - I)^2] \to 0$
We call this r.v. I the stochastic integral of g w.r.t. W on [0,t], and we denote it by $I = \int_0^t g(s)dW(s)$

Remark:

1) In the definition of I_n the term $W(t\frac{k+1}{n}) - W(t\frac{k}{n})$ is the increment of W over the interval $[\frac{tk}{n}, \frac{t(k+1)}{n}]$ exactly like in a Riemann-Stieltjes integral.

2) The convergence can't hold in a stronger sense (in particular we can not say that $\lim_n I_n$ exists)because W is not differentiable anywhere (keep in mind the picture of W)
In particular, W is not of bounded variation, i.e.

$$\sum_{k=0}^{n-1} |W(t\frac{k+1}{n}) - W(\frac{tk}{n})|_{n\to\infty} \longrightarrow \infty$$

with probability 1.(If W were differentiable, this sum$\to \int_0^t |\frac{dW(s)}{ds}(s)|$)

3) For the proof of proposition 1) (for those that want more mathematical rigorousness) usually what one tries when proving convergence is to ease the limit. But here we don't know the limit. We just call it "the stochastic integral". But since $\mathcal{L}^2(\Omega)$ is a Complete space (meaning that any Cauchy sequence is convergent), we just need to show that our sequence is Cauchy, i.e. $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall m, m \geq \mathbb{N}$, $E[(I_n - I_m)^2] < \epsilon$.

In general we will not use the definition of the stochastic integral, but rather its properties, that is why it is essential to known them:

Properties:

1) $E[\int_0^t g(s)dW(s)] = 0$

2) $E[\int_0^t g(s)dW(s)^2] = \int_0^t E[g(s)^2]ds$

3) For each $t \geq 0$, let $Y_t = \int_0^t g(s)dW(s)$. Then Y is adapted to $\mathcal{F}^W$.

Exercise: prove (3) by using the fact that $\mathcal{F}^W$ is right continuous, i.e. $\bigcap_{s\geq t} \mathcal{F}_s^W = \mathcal{F}_t^W$.

Theorem:

let $X_t = \int_0^t g(s)dW(s)$. Then X is a $\mathcal{F}^W$-martingale.

Remark:

1) We saw in last lecture that (X)-the stochastic integral, was an adapted process to $\mathcal{F}^W$.

2) To show that $E[X_t|\mathcal{F}_s^W] = X_s$, it is enough to show that $E[X_t - X_s|\mathcal{F}_s^W] = 0$ (since $X_s = E[X_s|\mathcal{F}_s^W]$), or equivalently $E[\int_s^t g(Z)dW(Z)|\mathcal{F}_s^W] = 0$

This property is not hard to prove, but it is a little tricky. We will show it just for simple function g, i.e. functions that are constant over intervals.(* The general case for g is proved by approximating g with such simple functions)

Exercise: Show directly that a B.m. W is a martingale w.r.t. its own $\mathcal{F}^W$.

Theorem:

Let M be a martingale such that for any $t \geq 0$, $E[(M_t)^2] < \infty$ (i.e. square integrable martingale)

The quantity $\sum_{k=0}^{n-1}[M(\frac{(k+1)t}{n}) - M(\frac{kt}{n})]^2$ converges in $\mathcal{L}^2(\Omega)$ to a r.v. $A(t)$ that is called quadratic variation of M, and it is denoted by $A(t) =< M > (t)$.

Properties:

1) A is a stochastic process that is adapted to $\mathcal{F}^M$, i.e. $A(t) \in \mathcal{F}_t^M$.

2) $A(t)$ is non-decreasing a.s.

Remark:

For B.m. we have

$$E[\sum_{k=0}^{n-1}[B(\frac{(k+1)t}{n}) - B(\frac{kt}{n})]^2]$$

$$= \sum_{k=0}^{n-1} E[[B(\frac{(k+1)t}{n}) - B(\frac{kt}{n})]^2] = \sum_{k=0}^{n-1} \frac{t}{n} = t$$

Exercise: Show that $< B > (t) = t$, $< W > (t) = t$, if B is a B.m.

Remark:

The converse is also true (Poul Levy): If M is a square integral martingale that is a.s. continuous, and if $< M > (t) = t$, then M is a B.m.

Interpretation:

What is the interpretation of quadratic variation ?

$$(*) \sum_{k=0}^{n-1} |f(\frac{t(k+1)}{n}) - f(\frac{tk}{n})|^{1+\alpha} \to ? \ as\ n \to \infty$$

For B.m. $(*) \to \infty$ if $\alpha < 1$; $(*) \to t$ if $\alpha = 1$: $(*) \to 0$ if $\alpha > 1$

When $\alpha = 0$, (*) should converge to $\int_0^t |g'(s)|ds$.But B.m. is to squiggly. It is not differentiable any where. So we would like to ask ourselves how non differentiable the function is. So we increase α until we can differentiate. It is related with the fractal structure.

* It measures the variation of the process, if the process does not have derivatives.

Definition

The joint variation of two square integrable martingales X, Y is defined y the following formula:

$$< X, Y >= \frac{< X + Y > - < X > - < Y >}{2}$$

Remark:

It turns out that

$$< X, Y >= \lim_{n\to\infty} \sum_{k=0}^{n-1} [X(\frac{(k+1)t}{n}) - X(\frac{kt}{n})] Y(\frac{(k+1)t}{n}) - Y(\frac{kt}{n})]$$

Definition

(semimartingales) let $W(t)$ be B.m. and let $\{\mu(t)\}_{t\geq 0}$,$\{\sigma(t)\}_{t\geq 0}$ be $\mathcal{F}^W-$adapted stochastic process. Let $X(t) = X(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s)$. This process is not a martingale (it would be if $\mu \equiv 0$). We call this process a (continuous, square integrable) semimartingale w.r.t. $\mathcal{F}^W$.

Example

Any martingale plus a drift μ will give us a semimartingale. If the drift is positive then the semimartingale is a submartingale (show that.)

Proposition

A semimartingale has a quadratic variation too, i.e. the limit

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}[X(\frac{(k+1)t}{n}) - X(\frac{kt}{n})]^2 \ \ exists\ in\ \mathcal{L}^2(\Omega)$$

Moreover,

$$< X > (t) = \int_0^t \sigma(s)^2 ds =< M > (t).$$

where $M(t)$ is the martingale part of the semimartingale, i.e. $M(t) = \int_0^t \sigma(s)dW$.

Ito^n formula

Let X be a semimartingale defined by

$$dX(t) = \mu(t)dt + \sigma(t)dB(t)$$

and let $f \in C^2$. then

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX(s) + \frac{1}{2}\int_0^t f''(X(s))d < X > (s)$$

$$= f(X(0)) + \int_0^t [f'(X(s))\mu(s) + \frac{1}{2}f''(X(s))\sigma^2(s)]ds + \int_0^t f'(X(s))\sigma(s)dB(s)$$

In particular, if $X(t) = B(t)$ (B.m.) then

$$f(B(t)) = f(B(0)) + \int_0^t f'(B(s))dB(s) + \frac{1}{2}\int_0^t f''(B(s))d < B > (s)$$

Sketch of proof ?

In conclusion, remember $(dB(s))^2 = ds$, $d < X > (s) := (dX(s))^2 = \sigma^2(s)ds$, $(dt)^2 = 0$