Lecture 8

Math 50051, Topics in Probability Theory and Stochastic Processes

Recall that a sequence x_n of real numbers is said to converge to a limit x, as $n \to \infty$, if $|x_n - x| \to 0$, as $n \to \infty$. Clearly, when we consider X_n with distribution $F_n(x)$, the existence of a limit X with distribution $F(x)$ must depend on the properties of the sequences $|X_n - X|$, and $|F_n(x) - F(x)|$. We therefore define the events

$$A_n(\epsilon) = \{|X_n - X| > \epsilon\}, \quad \textit{where } \epsilon > 0$$

Summation lemma. This gives a criterion for a type of convergence called *almost sure convergence*. It is straightforward to show that, as $n \to \infty$,

$$P(X_n \to X) = 1,$$

if and only if finitely many $A_n(\epsilon)$ occur, for any $\epsilon > 0$.

Convergence in probability. If, for any $\epsilon > 0$,

$$P(A_n(\epsilon)) = P(|X_n - X| > \epsilon) \to 0, \quad \textit{as } n \to \infty,$$

then X_n is said to *converge in probability* to X. We may write $X_n \xrightarrow{P} X$.

It is trivial to see that almost sure convergence implies convergence in probability; formally

$$X_n \xrightarrow{a.s.} X \Longrightarrow X_n \xrightarrow{P} X.$$

Convergence in mean square. From Chebyshov's inequality we have

$$P(|X_n - X| > \epsilon) \leq E|X_n - X|^2/\epsilon^2.$$

Therefore, if we can show that $E|X_n - X|^2 \to 0$ as $n \to \infty$, it follows that $X_n \xrightarrow{P} X$. This is often a very convenient way of showing convergence in probability, and we give it a name: if $E|X_n - X|^2 \to 0$ as $n \to \infty$, then X_n is said to *converge in mean square* to X. We may write $X_n \xrightarrow{m.s.} X$.

However, even this weaker form of convergence sometimes fails to hold; in the last resort we may have to be satisfied with showing convergence of the distributions $F_n(x)$. This is a very weak form of convergence, as it does not even require the random variables X_n to be defined on a common probability space.

Convergence in distribution. If $F_n(x) \to F(x)$ at all the points x such that $F(x)$ is continuous, then X_n is said to *converge in distribution*. We may write $X_n \xrightarrow{D} X$.

Stochastic processes. A family of random variables $(X_t)_{t \in T}$ is called a stochastic process. They are typically used as a mathematical model of the outcomes of a series of random phenomenon, such as the value of the IBM stock, a certain option price, for example, at time t. If T is a discrete set then the stochastic processes are called discrete stochastic processes. In perticuler, if $T = \{0, 1, 2, ...\}$ or $T = \{1, 2, ...\}$ then we are talking about discrete-time stochastic processes. In this case the random variable $X_1, X_2, ...$ can record the IBM stock price on consecutive business days. The prices might not be evenly space out (i.e. if X_1 is the price on Thursday, X_2 on Friday then X_3 is the price on Monday), the counting $1, 2, ...$ refers only at the order of the prices.

When T is an interval in $\mathbb{R}$ (typically $T = [0, \infty)$), we shall say that $(X_t)_{t\in[0,\infty)}$ is a stochastic process in continuous time.

If $\omega \in \Omega$ is fixed then the function

$$t \to X_t(\omega) = X(t, \omega)$$

is called a sample path.

Observe that when X is in discrete time, the sample path is the sequence $X_1(\omega), X_2(\omega), ...$

Example: A classic example of a stochastic process is the one where we consider a particle that, at time 0, is at the origin. At each time unit, a coin is tossed. If "tails" (respectively, "heads") is obtained, the particle moves one unit to the right (resp., left). Thus, the random variable X_n denotes the position of the particle after n tosses of the coin, and the s.p. $\{X_n, n = 0, 1, ...\}$ is a particular *random walk*. Note that here the index n can simply denote the toss number (or the number of times the coin has been tossed) and it is not necessary to introduce the notion of *time* in this example.

Example: An elementary continuous-time s.p., $\{X(t), t \geq 0\}$, is obtained by defining

$$X(t) = Yt \quad for \;\; t \geq 0$$

where Y is a random variable having an arbitrary distribution.

The set $V_{X(t)}$ of values that the rvs $X(t)$ can take is called **state space** of the stochastic process $\{X(t), t \in T\}$. If $V_{X(t)}$ is finite or countably infinite (resp uncountably infinite) then $\{X(t), t \in T\}$ is said to be a **discrete-state** (resp., **continuous-state**) process.

In the examples above the random walk is a discrete time and a discrete space sp, while the continuous time process is a continuous space process, unless Y takes the value 0.

Filtrations. As the time t increases, we have more and more knowledge about our stock prices, about what happened in the past. Let's think at the knowledge that we have at time t as $\mathcal{F}_t$ — a σ-field from $\mathcal{F}$. Then because our knowledge increases, it is natural to have

$$\mathcal{F}_s \subseteq \mathcal{F}_t, \quad if \; s \leq t$$

A family $\mathcal{F}_t$ of σ-fields on Ω (included in the absolute knowledge $\mathcal{F}$) is called a filtration, if

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$$

for any $s, t \in T$ such that $s \leq t$.

Remark. $\mathcal{F}_t$ contains all events A such that at time t we can decide if A has occured or not.

Example: 1) Let $X_1, X_2, ...$ be a sequence of coin tosses and $\mathcal{F}$ be the σ-field generated by $X_1, X_2, ..., X_n$. Let A={ the first 6 tosses produce at least 4 tails}. Is $A \in \mathcal{F}_4$? $A \in \mathcal{F}_5$? $A \in \mathcal{F}_6$? $A \in \mathcal{F}_{10}$?

We say that the process $(X_t)_{t\in T}$ is **adapted** to the filtration $(\mathcal{F}_t)_{t\in T}$, if X_t is $\mathcal{F}_t$-measurable for each $t \in T$. In other words, if the values of X_t are known given the information $\mathcal{F}_t$ then the processes is adapted.

If $\mathcal{F}_t$ is given, do I know X_s, $s < t$?

Remark: Given a stochastic process X_t the sequence of sigma algebras $\mathcal{F}_t^X = \sigma\{X_s, s \leq t\}$ forms a filtration. Indeed, if $s \leq t$, $\mathcal{F}_s \subseteq \mathcal{F}_t$.

Example: 1) Is $A = \{X(s) > 5 \text{ for all } s \leq 9\} \in \mathcal{F}_9^X$?

2) Is $A = \{X(s) > 6 \text{ for some } s \leq 10\} \in \mathcal{F}_{10}^X$?

3) Is $E = \int_0^3 [X(s)^{32} + cos 2\pi s]ds \in \mathcal{F}_3^X$?

4) Is $M_t = sup_{s\leq t}\, X(s) \in \mathcal{F}_t^X$? So, is M adapted to $\mathcal{F}^X$?

5) Is $N_t = inf_{s\geq t}|X(s)| \in \mathcal{F}_s^X$? Is N adapted to $\mathcal{F}^X$? Is $N_t \in \mathcal{F}_t^X$?

6) Is $L = lim_{s\to\infty} inf|X(s)| \in \mathcal{F}_t^X$ for some t ? $\in \mathcal{F}_s^X$?

Definition: If the random variable $X(t_4) - X(t_3)$ and $X(t_2) - X(t_1)$ are independent for any $t_1 < t_2 < t_3 < t_4$, we say that the stochastic process $\{X(t), t \in T\}$ is a process with **independent increments.**

Definition: If the random variable $X(t_2+s)-X(t_1+s)$ and $X(t_2)-X(t_1)$ have the same distribution function for all s, $\{X(t), t \in T\}$ is said to be a process with **stationary increments**.

Remarks: The random variables $X(t_2+s)-X(t_1+s)$ and $X(t_2)-X(t_1)$ in the preceding definition are *identically distributed*. However, in general, they are *not* equal.

Example: *Independent* trials for which the probability of *success* is the same for each of these trials are called *Bernoulli trials*. For example, we can roll some die independently an indefinite number of times and define a success as being the rolling of a "6".

A *Bernoulli process* is a sequence $X_1, X_2, ...$ of Bernoulli r.v.s associated with Bernoulli trials. That is, $X_k = 1$ if the kth trial is a success and $X_k = 0$ otherwise. We easily calculate

$$E[X_k] = p \quad \forall k \in \{1, 2, ...\}$$

where p is the probability of a success.

Definition: We say that the stochastic process $\{X(t), t \in T\}$ is **stationary**, or **strict-sense stationary** (SSS), if its distribution function of order n is invariant under any change of origin:

$$F(x_1, ..., x_n; t_1, ..., t_n) = F(x_1, ..., x_n; t_1 + s, ..., t_n + s)$$

for all s, n, and $t_1, ..., t_n$.

Remarks: The value of s in the preceding definition must be chosen so that $t_k + s \in T$, for $k = 1, ..., n$. So, if $T = [0, \infty)$, for instance, then $t_k + s$ must be nonnegative for all k.

Example: An elementary example of a strict-sense stationary stochastic process is obtained by setting

$$X(t) = Y \quad for \;\; t \geq 0$$

where Y is an arbitrary random variable. Since $X(t)$ does not depend on the variable t, the process $\{X(t), t \geq 0\}$ necessarily satisfies the equation in the definition of SSS.