Lecture 9

Math 50051, Topics in Probability Theory and Stochastic Processes

More examples of MC

1) Let $S = \mathbb{Z}$ and (X_n) be a sequence of iid random variables with $P(X_1 = 1) = p$ and $P(X_1 = -1) = 1 - p$ (X_n tells you if you take a step to the left or to the right at time $t = n$). Define $Y_n = \sum_{i=1}^n X_i$ (Y_n tells you where you are at time $t = n$) and let $Y_0 = 0$. Show that Y_n is a Markov Chain and find its transitional probabilities. Y_n is called a random walk on $\mathbb{Z}$ starting at 0. Replacing $Y_0 = 0$ with $Y_0 = i$, we get a random walk that starts at i.

2) If gambler wins \$1 with probability p and loses \$1 with probability $1 - p$ at each play of the game. He quits when he goes broke or he attains a fortune of \$$N$. Let X_n be the gambler's fortune at time n. Then what can you say about $\{X_n, n \geq 0\}$? $\{X_n, n \geq 0\}$ is also called a random walk but on $S = \{0, 1, ..., N\}$ absorbed at 0 and N (absorbing barriers).

Transition probabilities in n steps Consider the case when the process moves from state i to state j in n steps. Of course, a first question to ask is what is the probability that a MC that is in state i will be in state j after n additional transitions. These transition probabilities in n steps, denoted by $p_{i,j}^{(n)} = P(X_{m+n} = j | X_m = i)$, for $m, n, i, j \geq 0$ are recorded in a matrix denoted by $P^{(n)}$, named **n-step transition matrix**. Observe that these probabilities do not depend on m, ie they are the same for any m. Indeed,

$$p_{i,j}^{(n)} = P(X_{m+n} = j | X_m = i) = P(X_n = j | X_0 = i),$$

because of the time homogeneity.

Turns out these matrices have some nice properties:
Theorem: Chapman-Kolmogorov equation Suppose that X_n is an $S-$valued Markov chain with $n-$step transition probabilities $p_{i,j}^{(n)}$. Then for all $k, n \in N$ we have

$$p_{i,j}^{(m+n)} = \sum_S p_{i,s}^{(m)} p_{s,j}^{(n)}, \quad \text{for all } i, j \in S$$

where $p_{i,j}^{(0)} = 1$ if $i = j$ and is 0 otherwise.
Remark: This implies that $P^{(m+n)} = P^{(m)} P^{(n)}$, and even more

$$P^{(n)} = P^n$$

Remark: If we are interested in unconditional probabilities associated to MC then we need to know the initial distribution, $\pi^{(0)}$, because

$$P(X_n = j) = \sum_{s \in S} p_{s,j}^{(n)} \pi_s^{(0)}.$$

Example: Suppose that if it rains today then it'll rain tomorrow with probability .7 and if it doesn't then it'll rain tomorrow with probability .4. What's the probability it will rain two days from now given that is raining today? If we know that the probability that will rain today is .4, what is the probability that it will rain in two days from now?

Classes and states We say state j is **accessible** from state i or that state i **communicates** with state j if $p_{i,j}^{(n)} > 0$ for some $n \geq n$, ie the chain will visit state j if it starts in state i in n steps, for some n positive. Two states i and j **intercommunicate** if they are both accessible or if both communicate to each other.

Notation:
$i \to j$ j is accessible, or i communicates with j.
$i \leftrightarrow j$ i and j intercommunicate.

Remark: If two states do not intercommunicate then either

$$p_{i,j}^{(n)} = 0 \quad \text{for all } n \geq 0$$

or

$$p_{j,i}^{(n)} = 0 \quad \text{for all } n \geq 0$$

Properties: Intercommunication is an equivalence relationship, meaning it verifies the following 3 properties

1) It is reflexive, ie $i \leftrightarrow i$.
2) It is symmetric, ie $i \leftrightarrow j$ implies $j \leftrightarrow i$;
3) It is transitive, ie, if $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$.
Why is this true?

Two states that intercommunicate are said to be in the same **class**. If **all** states intercommunicate then we say that the MC is **irreducible**. If there are several classes in the MC we say the MC is reducible.

Examples:

Notation Denote by $f_{ii}^{(n)} = P[X_n = i, X_m \neq i, m = 1, 2, ..., n-1 | X_0 = i]$ – the probability that starting from i, the first return to i occurs at the n'th transition and by $f_{ii} = \sum_{n=0}^{\infty} f_{ii}^{(n)}$ – the probability that starting from i, the process will ever return to i. ($f_{ii}^{(0)} = 0$).

Definition State i is said to be recurrent iff $f_{ii} = 1$; State i is said to be transient iff $f_{ii} < 1$.

Consequences:

c1) State i is recurrent implies that starting in i, the process will revisit i infinitely many times with probability 1 (the M.P. and time homogeneity implies this.)

c2) i transient $\Longrightarrow P[X \; will \; ever \; return \; i | X_0 = i] = f_{ii}$

$\Longrightarrow P[X \; will \; be \; in \; i \; for \; exactly \; n \; time \; epochs | X_0 = i] = f_{ii}^{n-1}(1 - f_{ii})$. **Note** it is f_{ii}^{n-1} not $f_{ii}^{(n-1)}$.

$\Longrightarrow$ Given $X_0 = i$, the number of time epochs in state i is distributed $Geom(1 - f_{ii})$.

$\Longrightarrow E[number\ of\ times\ epochs\ in\ i|X_0 = i] = \frac{1}{1-f_{ii}} < \infty$.

$\Longrightarrow P[X\ revisits\ i\ \infty\ times\ |X_0 = i] = 0$.

Proposition: Let $X = \{X_n\}$ be a $M.C$ with state space S. Let $i \in S$, then
If $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$, then i is recurrent.
If $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$, then i is transient.

Why?

Corollary 1 In a finite state $M.C.$ not all states can be transient.

Proof: Assume $S = \{1, 2, ..., N\}$ and suppose all states are transient. Let T_i = the time after which state i will never be revisited ($P[T_i < \infty] = 1, i \in S$). Then after time $T = max\{T_i\}$ no state will be revisited, with probability 1. But this is absurd.

Corollary 2: a) i recurrent & $i \longleftrightarrow j \Longrightarrow j$ is recurrent, and
i transient & $i \longleftrightarrow j \Longrightarrow j$ is transient.

b) All states in a finite irreducible $M.C.$ are recurrent.

Example 1

$$\begin{pmatrix} 0 & 0 & 1/3 & 2/3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

This is a finite irreducible $M.C.$, so all states are recurrent.

Example 2

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/2 \end{pmatrix}$$

reducible and we have 3 classes $\{0, 1\}, \{2, 3\}$ (recurrent), $\{4\}$ (transient).

Periodicity

Definition The period of state i is defined by $d(i) = g.c.d.\{n \geq 1; p_{ii}^{(n)} > 0\}$,where by g.c.d.we mean the greatest common divisor.

A state with period 1 is called aperiodic. Otherwise it is called periodic.

Fact: If i has period $d(i)$ and $i \longleftrightarrow j$, then j has period $d(i)$.

Example Let $X = \{X_n\}$ be a $M.C.$ with

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}, \quad d(0) = ?$$

Definition A recurrent state i is said to be positive recurrent if the expected time until the process returns to state i is finite. A recurrent state that is not positive recurrent is called null recurrent.

Remark:

I) The expected time until the process returns to state i is given by

$$m_i = \sum_{n=0}^{\infty} n f_{ii}^n$$

II) A recurrent state of a finite state $M.C.$ is positive recurrent state not null recurrent.

Definition A positive recurrent state that is aperiodic is called ergodic.

Fact: Intercommunication is an equivalency for transiency, recurrency, null-recurrency, positive recurrency, ergodicity and as mention above periodicity, i.e. if $i \longleftrightarrow j$ and i is transient, recurrent, null-recurrent, positive recurrent or aperiodic then so is j.

Regularity

Definition: let $\{X_n, n \geq 0\}$ be a $M.C.$ on S. The process is called regular if there is some $k > 0$ such that $P^k > 0$ (all of the elements of P^k are strictly positive, i.e. $p_{ij}^{(k)} > 0$, $\forall\ i, j \in S$).

Example 1 Consider the $M.C.$ with:

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 1/6 & 1/2 & 1/3 \end{pmatrix}$$

Example 2 Consider the $M.C.$ with:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 3 & 4 \\ 0 & 0 & 6 & 4 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Fact: If P is regular over N states then $P^{N^2} > 0$. (This means that if the $M.C.$ has N states and you computed all powers of P up to P^{N^2} and you did not find anything strictly positive then you are sure the $M.C.$ is not regular.)

Long-Time Behavior of the Markov Chains

Definition: Let $\{X_n\}_{n\geq 0}$ be a $M.C.$ with state space S and transition matrix P. Suppose that for all $i, j \in S$

$$\lim_{n\to\infty} p_{ij}^{(n)} = \Pi_j, \quad \textit{the limit is independent of } i.$$

Then:

1) $\sum_j \Pi_j \leq 1$;

2) $\sum_i p_{ij}\Pi_i = \Pi_j$;

3) either $\sum_j \Pi_j = 1$ or $\Pi_j = 0$ for all $j \in S$.

Proof: First let's assume S is finite with m elements.

$$\sum_{j\in S} \Pi_j = \sum_{j=1}^{m} \Pi_j = \sum_{j=1}^{m} \lim_{n\to\infty} p_{ij}^{(n)} = \lim_{n\to\infty} \sum_{j=1}^{m} p_{ij}^{(n)} = \lim_{n\to\infty} 1 = 1$$

This proves 1) and 3).

To prove 2), let us fix $j \in S$ and $k \in S$. Then

$$\sum_{i=1}^{n} p_{ij}\Pi_i = \sum_{i=1}^{n} \lim_{n\to\infty} p_{ij}\tau_{ki}^{(n)} = \lim_{n\to\infty} \sum_{i=1}^{m} p_{ij}\tau_{ki}^{(n)} = \lim_{n\to\infty} \sum_{i=1}^{m} p_{ki}^{(n)} p_{ij} = \lim_{n\to\infty} \tau_{ki}^{(n+1)} = \Pi_j.$$

when S is infinite (but still countable) the issue is of interchanging lim and $\sum$ which is not in general true. One can use a Fatou lemma argument, but it is not the object of our study. We consider it.

Fact: If $X = \{X_n, n \geq 0\}$ is a regular $M.C.$ on the state space $S = \{0, 1, ..., N\}$,then there exists a limiting probability distribution

$$\Pi = (\Pi_0, \Pi_1, ..., \Pi_N)$$

From property 3) of the above proposition $\prod_{i\in S} \Pi_j = 1$ and from property 2) we deduce

$$\Pi = \Pi P$$

Hence the limiting distribution of a regular $M.C.$ on a finitely dimensional state space is the solution of the equation

$$\Pi = \Pi P$$

$$\sum_{j=0}^{N} \Pi_j = 1.$$

Definition A probability $\mu = \sum_{j\in S} \mu_j \delta_j$, with $\mu_j \geq 0$ is an invariant measure of a $M.C.$ $(X_n)_{n\in\mathbb{N}}$ with transition matrix P, if for all $n \in \mathbb{N}$, $j \in S$,

$$\sum_{i\in S} p_{ij}\mu_i = \mu_j$$

(i.e. $(\mu_i)_{i\in S}$ is a left eigenvector of P with eigenvalue 1.) Here δ_j is the Dirac measure. i.e. $\delta_j(A) = \begin{cases} 1 & if\ j \in A \\ 0 & if\ j \notin A \end{cases}$

Remark: If $\Pi = (\Pi_j)_{j\in S}$ is the limiting distribution from above, then $\mu = \sum_{j\in S} \Pi_j \delta_j$ is the unique invariant measure of our $M.C.$

Theorem Suppose that S is finite and the transition matrix P of a $M.C.$ on S satisfies:

$$J_{n_0} \in \mathbb{N},\ \ J_\epsilon > 0 : \rho_{ij}^{(n_0)} \geq \epsilon,\ \ i, j \in S$$

Then the following limit exists for all $i, j \in S$ and is independent of i:

$$\lim_{n\to\infty} p_{ij}^{(n)} = \Pi_j$$

where the numbers Π_j satisfy:

$$\Pi_j > 0, j \in S \quad and \quad \sum_{j\in S} \Pi_j = 1$$

Corollary:

A regular MC on a finite state space S satisfies the conditions from the above theorem. Therefore for such a MC There exists a limiting probability π, verifying:

$$\pi_j > 0,\ \ j \in S$$

$$\sum_{j\in S} \pi_j = 1$$

$$\sum_{i\in S} p_{ij}\pi_i = \pi_j$$

This limiting probability defines the unique invariant measure

$$\mu = \sum_{i\in S} \pi_i \delta_i$$

Example 1

$$P = \begin{pmatrix} 0.4 & 0.5 & 0.1 \\ 0.05 & 0.7 & 0.25 \\ 0.05 & 0.5 & 0.45 \end{pmatrix}$$

find Π.

Example 2 Let $\{X_n\}$ be a $M.C.$ with $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_{2\times 2}$, Obviously

$$P^n = P, \forall n \ \Rightarrow \lim_{n\to\infty} P^n = P$$

So $\{X_n\}$ has a limiting distribution but it obviously depends on the initial state:

— If it starts in 0 $\Rightarrow \Pi = (1, 0)$

— If it starts in 1 $\Rightarrow \Pi = (0, 1)$

Example 3 Let $\{X_n\}$ be a $M.C.$ with

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\{X_n\}$ oscillates almost surely between the two states.

$\{X_n\}$ is periodic (no-limiting distribution)

$$P^n = \begin{cases} P & n\ odd \\ I_{2\times 2} & n\ even \end{cases}$$

Suppose that $(X_n)_{n \in mathbbN}$ is a $M.C.$ with state spaces S. Let $j \in S$ be a recurrent state

I) If j is aperiodic, then $P_{jj}^{(n)} \to \frac{1}{m_j}$ Moreover, for $i \in S$, $P_{ij}^{(n)} \to \frac{F_{ij}^{(1)}}{m_j}$ where $F_{ij}^{(1)}$ is the probability that the chain will ever visit state j, if it starts at i and m_j is the mean recurrence time of state j.

II) If j is a periodic state of period $d \geq 2$, then $P_{jj}^{(nd)} \to \frac{d}{m_j}$

Corollary Let $\{X_n\}$ be a finite irreducible aperiodic (ergodic) $M.C$, then

(i) X is regular

(ii) there exists a limiting probability Π satisfying

$$\Pi = \Pi P \quad and \quad \sum_{k \in S} \Pi_k = 1 \qquad *$$

(iii) $\Pi_j = \frac{1}{m_j}$

Remarks:

1) Any vector Π satisfying (*) is called a stationary probability distribution vector.

2) A limiting distribution Π, when it exists, is always a stationary distribution, but the converse is false.

Example Let $\{X_n\}$ be a $M.C.$ with $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $\Pi = (1/2, 1/2)$ is a stationary distribution for $\{X_n\}$, since $(1/2, 1/2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (1/2, 1/2)$, but we saw $\{X_n\}$ has no-limiting distribution.

Example P is doubly stochastic iff the sum over each column (as well as each row) is 1. If such a chain is irreducible and aperiodic over $S = \{0, 1, ..., N\}$, then the limiting probabilities are given by

$$\Pi_j = \frac{1}{N+1}, \quad \forall\, j \in S.$$

Example A particle moves on a circle through points marked 0,1,2,3,4 (clockwise). At each step has probability p to move right and $1-p$ to move left. Let X_n be the particle location on the circle after the n-th step. Calculate the limiting probability for $\{X_n\}$.

Theorem

I) Let $\{X_n\}$ be a $M.C.$ on a state space S, then there exists an invariant measure if and only if each recurrent state is positive-recurrent. Moreover, in this case, the invariant measure is unique and it is given by

$$\mu = \sum_i \frac{1}{m_i} \delta_i.$$

II) Suppose $\{X_n\}$ is an irreducible aperiodic $M.C.$ with state space S, then $\{X_n\}$ is ergodic is and only if it has a unique invariant measure.