$$
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix}
0 & 1/3 & 2/3 & 0 & 0 \\
0 & 0 & 0 & 1/4 & 3/4 \\
0 & 0 & 0 & 1/2 & 1/2 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

1. $\{1, 2, 3, 4, 5\}$ one class irreducible. $\Longleftrightarrow$ intercommunicate.

2. period: 1 to 1, greatest common divisor — 3.
   the states in one class have the same period

3. $P_{21}^{1000} = 1$. from 2 to 1, in 1000 steps

   $P_{22}^{1000} =$ $\Big\}$ read from P^{1000}

   $P = QDQ^{-1}$

   $P^{1000} = QD^{1000}Q^{-1}$

$$
\det(\lambda I - A) = 0 \qquad
\begin{bmatrix}
\lambda & -1/3 & -2/3 & 0 & 0 \\
0 & \lambda & 0 & -1/4 & -3/4 \\
0 & 0 & \lambda & -1/2 & -1/2 \\
-1 & 0 & 0 & \lambda & 0 \\
-1 & 0 & 0 & 0 & \lambda
\end{bmatrix}
= (-1)\begin{bmatrix}
-1/3 & -2/3 & 0 & 0 \\
\lambda & 0 & -1/4 & -3/4 \\
0 & \lambda & -1/2 & -1/2 \\
0 & 0 & \lambda & 0
\end{bmatrix}
+ \lambda \begin{bmatrix}
\lambda & -1/3 & -2/3 & 0 \\
0 & \lambda & 0 & -1/4 \\
0 & 0 & \lambda & -1/2 \\
-1 & 0 & 0 & \lambda
\end{bmatrix}
$$

$$
= (-1)(-\lambda)\begin{bmatrix}
-1/3 & -2/3 & 0 \\
\lambda & 0 & -3/4 \\
0 & \lambda & -1/2
\end{bmatrix} + \lambda \cdots
$$

find eigenvalue, Q

$$
P^2 = \begin{bmatrix}
0 & 1/3 & 2/3 & 0 & 0 \\
0 & 0 & 0 & 1/4 & 3/4 \\
0 & 0 & 0 & 1/2 & 1/2 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1/3 & 2/3 & 0 & 0 \\
0 & 0 & 0 & 1/4 & 3/4 \\
0 & 0 & 0 & 1/2 & 1/2 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & \frac{5}{12} & \frac{7}{12} \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1/3 & 2/3 & 0 & 0 \\
0 & 1/3 & 2/3 & 0 & 0
\end{bmatrix}
$$

$$
P^3 = \begin{bmatrix}
0 & 0 & 0 & 5/12 & 7/12 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1/3 & 2/3 & 0 & 0 \\
0 & 1/3 & 2/3 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1/3 & 2/3 & 0 & 0 \\
0 & 0 & 0 & 1/4 & 3/4 \\
0 & 0 & 0 & 1/2 & 1/2 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1/3 & 2/3 & 0 & 0 \\
0 & 1/3 & 2/3 & 0 & 0 \\
0 & 0 & 0 & 5/12 & 7/12 \\
0 & 0 & 0 & 5/12 & 7/12
\end{bmatrix},
\quad
P^4 = \begin{bmatrix}
0 & 1/3 & 2/3 & 0 & 0 \\
0 & 0 & 0 & 5/12 & 7/12 \\
0 & 0 & 0 & 5/12 & 7/12 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

find the pattern

T = the first return time to 1 starting at 1.

$\pi P = \pi$. π is the eigenvector of 1. $\pi = [\pi_1, \pi_2, \pi_3, \pi_4, \pi_5]$

$$\begin{cases} \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1 \leftarrow \text{sum } \pi_i = 1 \\ \pi_4 + \pi_5 = \pi_1 \\ 1/3\, \pi_1 = \pi_2 \\ 2/3\, \pi_1 = \pi_3 \\ 1/4\, \pi_2 + 1/2\, \pi_3 = \pi_4 \\ 3/4\, \pi_2 + 1/2\, \pi_3 = \pi_5 \end{cases} \qquad \begin{cases} \pi_1 = 1/3 \\ \pi_2 = 1/9 \\ \pi_3 = 2/9 \\ \pi_4 = 5/36 \\ \pi_5 = 7/36. \end{cases}$$

$\pi_1 = \frac{1}{3} = \lim_{n\to\infty} P_{i1}^{(n)}$ for any i

use π to find the first time to 2 starting at 2.

$$E(T) = \frac{1}{\pi_i} = \frac{1}{\pi_2} = 9.$$

Ex: (lecture 8)

1. $\begin{pmatrix} 0 & 0 & 1/3 & 2/3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ — all states in one class. → irreducible.

state 1. could go back to state 1 infinitely times → recurrent.
all states in a finite irreducible M.C. are recurrent

2. $\begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/2 \end{pmatrix}$

class: $\underbrace{\{0,1\}}_{\text{recurrent}}$. $\underbrace{\{2,3\}}_{\text{recurrent}}$. $\underbrace{\{4\}}_{\text{transient}}$.

Ex: lecture 9

1. $\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 1/2 & 0 & 1/2 & 0 \end{array}$, period ?

all states in one class, so consider one of them is enough.
$P(0)$, from 1 to 1. greatest common divisor.
⇓
in 4 steps, in 6 steps, 8, 10, ...
so, $P(0) = 2$.

2. Regular. $P^k > 0$. (each entry > 0)

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 3 & 4 \\ 0 & 0 & 6 & 4 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. $\Rightarrow P^2 = \begin{pmatrix} 5 & 8 & 27 & 24 \\ 4 & 9 & 27 & 24 \\ 4 & 0 & 36 & 24 \\ 1 & 2 & 3 & 4 \end{pmatrix} \Rightarrow P^3 > 0$. so regular.

only one is 0.

Ex1:

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & & \\ 1 & & \end{array} \quad \begin{bmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{bmatrix}$$

at state 0 when $t=0$.

what's the probability at state 1 when $t=3$.

$\Downarrow$

$P_{01}^{(3)} \rightarrow$ read from P^3

$$P^3 = \begin{bmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{bmatrix}$$

Ex 2:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.6 & 0 & 0.4 \\ 0.2 & 0.5 & 0.3 \end{bmatrix} \end{array}$$

long run. use two ways.

starts at 1, return 1

① find π_1, $\left(\pi_j = \lim_n P_{ij}^{(n)}\right)$, $\pi P = \pi$

② P^n. read P_{11}. $P^n = Q D^n Q^{-1}$

Ex 3.

$$\begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0.9 & 0 \\ 0.25 & 0.25 & 0 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0.7 & 0 & 0.3 & 0 \\ 0 & 0.2 & 0 & 0.2 & 0.2 & 0.4 \end{bmatrix}$$

1. $\underset{\text{recurrent}}{\{1, 2\}}$ $\underset{\text{recurrent}}{\{3, 5\}}$ $\underset{\text{transient}}{\{4, 6\}}$

2. at state 1, what's the probability at 1 in some large time? $P_{11}^{(n)}$, n large.

1,2 in one class, focus on $\begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}$. find eigenvalues. Q, D

$$d(\lambda I - P) = d\begin{pmatrix} \lambda-0.5, & -0.5 \\ -0.3, & \lambda-0.7 \end{pmatrix} = (\lambda - \tfrac{1}{2})(\lambda - \tfrac{7}{10}) - \tfrac{3}{10}\cdot\tfrac{1}{2} = \lambda^2 - \tfrac{6}{5}\lambda + \tfrac{7}{20} - \tfrac{3}{20} = \lambda^2 - \tfrac{6}{5}\lambda + \tfrac{1}{5} = (\lambda - \tfrac{1}{5})(\lambda - 1) = 0$$

$\lambda = \frac{1}{5}$, $\lambda = 1$ $\qquad D = \begin{bmatrix} 1/5 & 0 \\ 0 & 1 \end{bmatrix}$

$Ax = \frac{1}{5}x$

$$\begin{bmatrix} 0.5x_1 + 0.5x_2 \\ 0.3x_1 + 0.7x_2 \end{bmatrix} = \begin{bmatrix} 0.2x_1 \\ 0.2x_2 \end{bmatrix} \Rightarrow 0.3x_1 + 0.5x_2 = 0 \Rightarrow x = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$Ax = 1 \cdot x \Rightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so, $Q = \begin{bmatrix} 5 & 1 \\ -3 & 1 \end{bmatrix}$

$Q^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 5 & 1 \\ -3 & 1 \end{bmatrix}$, $Q^{-1} = \begin{bmatrix} 1/8 & -1/8 \\ 3/8 & 5/8 \end{bmatrix}$

$5a - 3b = 1$ $\quad$ $5c - 3d = 0$

$3a + 3b = 0$ $\quad$ $3c + 3d = 3$

$a = \frac{1}{8},\ b = -\frac{1}{8}$ $\quad$ $c = \frac{3}{8},\ d = \frac{5}{8}$

$$QDQ^{-1} = \begin{bmatrix} 5 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1/5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/8 & -1/8 \\ 3/8 & 5/8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3/5 & 1 \end{bmatrix} \begin{bmatrix} 1/8 & -1/8 \\ 3/8 & 5/8 \end{bmatrix}$$

Check

$$= \begin{bmatrix} 1/2 & 1/2 \\ 3/10 & 7/10 \end{bmatrix} = P \checkmark$$

so $P^n = QD^nQ^{-1} = \begin{bmatrix} 5 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} (1/5)^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/8 & -1/8 \\ 3/8 & 5/8 \end{bmatrix}$

$$= \begin{bmatrix} (1/5)^{n-1} & 1 \\ -3(1/5)^n & 1 \end{bmatrix} \begin{bmatrix} 1/8 & -1/8 \\ 3/8 & 5/8 \end{bmatrix} = \frac{1}{8}(1/5)^{n-1} + \frac{3}{8}$$

entry 11

starts at 1 and goes back to 1 in some step.