Lecture 18

Arbitrage pricing in continuous time

We consider a market with 2 assets:

1) bank account: “risk-free asset” $B$. It has the dynamics:

$$\frac{dB(t)}{B(t)} = r(t)dt$$

Observe: no “$dW$” term! So,

$$B(t) = B(0)e^{\int_0^t r(s)ds}$$

This could be interpreted as a bank account with short interest rate $r$. If $r$ is deterministic then $B$ could be interpreted as the price of a bond.

2) One stock, whose price verifies:

$$dS(t) = S(t)\alpha(t, S(t))dt + S(t)\sigma(t, S(t))d\tilde{W}(t)$$

or:

$$\frac{dS(t)}{S(t)} = \alpha(t, S(t))dt + \sigma(t, S(t))d\tilde{W}(t)$$

represents the proportion of the charge in the stock price.

Here:
- $\alpha$ represents the mean rate of return of $S$
- $\sigma$ is the volatility of $S$.

Observe that indeed the stock price has a stochastic rate of return with mean $\alpha(t, S(t))dt$

L. Bachelier in 1900 (a student of Poincare) already had Black-Scholes pricing formula (but did not have it as a consequence of hedging which Black-Scholes did.)

**Black-Scholes model**

Assumptions: $r$, $\alpha$, $\sigma$ are deterministic, constants.

Question: What is wrong with this assumption?

**Definition**

As in the discrete case, European calls and puts are simple contingent claim. A contingent claim with maturity date $T$ (or a $T$-claim) is a stochastic r.v. $X$ that is $\mathcal{F}_T^S$ measurable ($X \in \mathcal{F}_T^S$), $X$ is a simple contingent claim if it depends only on $S_T$ (rather than $S_t$, $t \leq T$), i.e.

$$X = \Phi(S(T)),$$
**Remark**

So a contingent claim is a contract that assures that the holder gets $X$ at $T$. The requirement that $X \in \mathcal{F}_T$ means that at time $T$ it will be possible to determine the amount $X$.

**Remark**

A simple contingent claim is Markov (it depends only on the stock price at time $T$).

**Remark**

An American option: the buyer can decide to stop at any time $t \leq T$, then receiving $\Phi(S_t)$. European means that the decision is made at time $T$.

**Example**

1) Nonsimple claim: $X = \Phi(\{S(t) : t \leq T\})$

2) Asian options depends on $\int_0^T S(t)dt$ – the “average” of the stock price up to $T$.

Eg: call: $X = (S_T - \frac{1}{T} \int_0^T S(t)dt)^+$. Observe it depends on the path of the stock price.

The big question is what the holder of the option should pay for it (or the writer of the option should charge), i.e.: For a claim $X$ find $\pi(t, X)$, $\forall r \leq T$ if $X \in \mathcal{F}_T$. To answer this we need to assume the market is free of arbitrage. (there is no arbitrage portfolio)

**Recall**

An arbitrage portfolio is a self-financing portfolio such that:

1) $V(0) = 0$
2) $P(V(T) \geq 0) = 1$
3) $P(V(T) > 0) > 0$

A self financing portfolio verifies the equation:

$$\frac{dV(t)}{V(t)} = u_0(t) \frac{dB(t)}{B(t)} + u_1(t) \frac{dS(t)}{S(t)}.$$

**Proposition**

If $k$ is self-financing portfolio such that

$$dV(t) = k(t)V(t)dt$$

where $k$ is a stochastic process, then if $k$ is not an arbitrage we have

$$k(t) = r.$$

**Idea of pricing:**

Assumptions:
1) The derivative instrument in question is traded on the market.
2) The market is free of arbitrage.
3) The price process for the derivative asset is of the form
\[ \Pi(t, X) = F(t, S(t)), \]
where \( F \) is some smooth function.

Our goal is to determine \( |PI(T, X) \) if the above assumptions are met. This is how we will proceed:
1) All the parameters \( (\alpha, \sigma, \Phi, r) \) are given.
2) Form a self-financing portfolio using \( S \) and \( \pi(\cdot, X) \) as the assets in such a way that \( V(t) \) has no martingale part. 3) Since we have assumed no arbitrage it must be the case that \( k = r \).

This will give BLACK-SCHOLES PDE, a PDE with \( F \) as the unknown function.
\[
rF - rS \frac{\partial F}{\partial S} = \frac{\partial F}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \quad \forall t \geq 0 \quad \forall S \in R
\]
has terminal (boundary) condition: \( F(T, S(T)) = \Phi(S(T)) \)
4) The equation has an unique solution, thus giving us the unique pricing formula for the derivative, which is consistent with absence of arbitrage.

**Remark**
\( \alpha \) is gone from the PDE. This is the same phenomenon as in binomial model where the market measure \( Q \) is used instead of the objective prob. measure \( P \).

but the value of \( \alpha \) still comes into play in practice since \( \pi(t, X) = F(t, S(t)) \)

5) How do we solve B-S PDE? (We did it before, but let’s make some remarks)

First, one solve BS without the term \( rF \) and then multiply by \( e^{-\int_t^T rds} = e^{-r(T-t)} \), so
\[
F(t, s) = E_{t,s}[\Phi(S(T))|e^{-r(T-t)}]
\]
and its infinitesimal operator is
\[
(\mathcal{A}F)(x) = rx \frac{\partial F}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}
\]
hence \( X \) verifies
\[
dX(u) = rX(u)du + \sigma X(u)dW(u),
\]
with \( X(t) = s \)

This equation almost looks like the equation for \( S \) but has \( r \) instead. What happened?

What happened was that we don’t have the same B.m. \( \tilde{W} \). We have a different one \( W \) under a different probability measure: \( Q \). This is the “martingale measure” and under this measure \( S \) has the above dynamics! Of course this is an artificial measure of the market.

Why is this a martingale measure?
Under \( Q \) the discounted stock price \( S^*(t) = S(t)e^{-rt} \) is a martingale. Check this.