Lecture

Consider a fixed simple $T$-claim $\chi$, i.e. a claim of the form $\chi = \Phi(S_T)$ and assume we are standing at time $t$. If we buy $\chi$ and pay today (time $t$, $\pi(t, \chi)$), then

$$\pi(t, \chi) = F(t, S_t, T, \chi)$$

and

$$\pi(t, \chi) = E^Q_{t,s}[\chi e^{-\int_t^T r(\sigma) d\sigma}] = e^{-r(T-t)} E^Q_{t,x}[\chi]$$

There are two common variations of these.

**Forward contracts**

Consider a standard Black-Scholes model. A simple $F$-claim $\chi = \Phi(S_T)$ and assume we stand at time $t$. A **forward contract** on $\chi$, made at $t$, is a contract that stipulates that the holder of the contract pays the deterministic amount $K$ at the delivery time $T$, and receives the stochastic amount $\chi$ at $T$. Nothing is paid or received at time $t$.

We use the notation $K = f(t, T, \chi)$ and the question is what is $K$? We call $K$ the forward price of $\chi$, maturing at time $T$, calculated at time $t$.

Q: How do we find the price of this contract?
A: The entire contract is just a pay off: the buyer of the contract (holder) receives $\chi - K$. However this contract by definition has price 0 at time $t$

$$0 = \pi(t, \chi - K) = \pi(t, \chi) - \pi(t, K)$$

$$K = F(t, T, \chi) = e^{r(T-t)} \pi(t, \chi) = E^Q_{t,x} \Phi(S(T))$$

Q: What is the price of a forward contract at $t < t' < T$?
A:

$$\pi(t', \chi - K) = \pi(t', \chi) - e^{r(T-t)} \pi(t, \chi)e^{-r(T-t')}$$

$$= e^{-r(T-t')} [E^Q_{t', S(t')}[\Phi(S(T))] - E^Q_{t,S(t)}[\Phi(S(T))]] \neq 0$$

**Futures contracts**

A futures contract on $\chi$ is a contract for delivery of $\chi$ at $T$ but all the payments from the buyer (the holder) to the underwriter are made continuously over the time, such that if $F(t, T, \chi)$ is the futures price, the holder of the contract over the time interval $[s, s + \Delta s]$ receives

$$F(s + \Delta s, T, \chi) - F(S, T, \chi)$$

from the underwriter.

Finally the holder will receive $\chi$ and pay $F(T, T, \chi)$ at the delivery date $T$. 
The spot price, at any time \( t \) prior to delivery of obtaining the futures contract, is by definition 0. Hence
\[
F(u, T, \chi) = E^Q_{u, S(u)}[\chi].
\]
If the short rate is deterministic, the forward and future prices are equal.

**American options**

We call an option \( \chi \) an American option if it could be exercised at any time \( t \leq T \).

In general, an American option is the right to receive a random quantity \( Y(t) \) at any time \( t \leq T \), with the understanding that \( Y(t) \in \mathcal{F}_t^S \) (it could be determined at time \( t \) by knowing everything about \( s \) up to time \( t \)).

**Conclusion**

If \( Y = \{ Y(t) : t \leq T \} \) defines an American option, the price of \( Y \) at time \( t \leq T \), denoted by \( \pi(t, Y) \), is given by
\[
\pi(t, Y) = \sup_{\text{stopping time } \delta \in [t, T]} E^Q[e^{-r(\delta-t)}Y(\delta)].
\]

In an European case (exercise at \( T \)) the price of the option is
\[
\pi(t, Y) = E^Q[e^{-r(T-t)}Y(T)]
\]

or if we stop at \( \delta \) :
\[
\pi(t, Y) = E^Q[e^{-r(\delta-t)}Y(\delta)].
\]

Since the underwriter does not know when the option is exercise, he has to prepare for the worst and charge the largest price
\[
\sup_{\delta \in [t, T]} E^Q[e^{-r(\delta-t)}Y(\delta)].
\]

Solving this problem (calculating this supremum, and perhaps calculating a rule to determine an optional \( Y \)) typically involves solving a free boundary problem for PDE’s.

In general such problems are hard to solve and they don’t have a closed formula, but in the case of American call option, the problem is solved.

**Volatility**

In the Black-Scholes model
\[
\frac{dS(t)}{S(t)} = \alpha dt + \sigma dw(t)
\]
\[
B(t) = e^{rt}
\]
\( \alpha \) and \( \sigma \) must be estimated from past data. For the purpose of option pricing we just need \( \sigma \).

Theoretically since \( S^2\sigma^2 dt = d<S>_t \), we should have
\[
\sigma^2 = \frac{d<S>_t}{dt} \frac{1}{S(t)^2}.
\]
On the right hand side, in principle, everything is observable: $S(t) \in \mathcal{F}_t^S$, and $\forall \epsilon, < S > (t-\epsilon) \in \mathcal{F}_t^S \implies \frac{d<S>}{dt}(t) \in \mathcal{F}_t^S$.

**Recall**

$$\lim_{n \to \infty} \text{in } L^2(\Omega) \sum_{k=0}^{n-1} [S\left(\frac{(k+1)t}{n}\right) - S\left(\frac{kt}{n}\right)]^2 = < S > (t).$$

In practice calculating $\frac{d<S>}{dt}$ is impossible. Even calculating $< S > (u)$ for any $u \leq t$ is impossible. This is because we have knowledge of $S(u)$ at discrete times $u_0, u_1, ..., u_n = t$.

Therefore we will use an approximate calculation of $\sigma^2 = \frac{\frac{d<S>}{dt}S(t)}{S(t)^2}$ as follows:

Let $X(t) = ln(S(t))$. Remember

$$S(t) = S(0)e^{\sigma[w(t)-w(0)]+(r-\frac{1}{2}\sigma^2)t}$$

$$X(t) = lnS(t) = lnS(0) + \sigma[w(t) - w(0)] + (r - \frac{1}{2}\sigma^2)t$$

$$dX(t) = (r - \frac{1}{2}\sigma^2)dt + \sigma dw(t) \implies \frac{d < X > (t)}{dt} = \sigma^2$$

So $< X > (t) = \sigma^2 t$. Therefore, $\sigma^2 = \frac{1}{t} < X > (t)$.

$$\sigma^2 = \frac{1}{t} \lim_{n \to \infty} \text{in } L^2(\Omega) \sum_{k=0}^{n-1} [X\left(\frac{(k+1)t}{n}\right) - X\left(\frac{kt}{n}\right)]^2$$

An estimator for $\sigma^2$ can be taken by fixing a large $n$ in the above formula.

$$\sigma^2 = \frac{1}{t} \sum_{k=0}^{n-1} [X\left(\frac{(k+1)t}{n}\right) - X\left(\frac{kt}{n}\right)]^2 + O\left(\frac{1}{\sqrt{n}}\right)$$

$$\sigma^* = \sqrt{\frac{1}{t} \sum_{k=0}^{n-1} [\ln S\left(\frac{(k+1)t}{n}\right) - \ln S\left(\frac{kt}{n}\right)]^2}$$

is called historic volatility estimator.

**Question:**

* What if the assumption “$\sigma$ constant” is erroneous?

* How short an interval of time $[0,t]$ do you need to estimate $\sigma$ in that interval?

* If you think $\sigma$ varies with time, one could, for example, to use $\sigma$ estimated over one day as the estimate for the next day.

* When do you know that you need to change to a new volatility in your model?
Stochastic Volatility

The model is of the type:

\[
\frac{dS(t)}{S(t)} = \alpha dt + \sigma(Y_t)dw(t)
\]

where \(\sigma(\cdot)\) is a deterministic function (typically people take \(\sigma(Y) = e^Y\)), and \(\{Y(t), t \geq 0\}\) is an adapted stochastic process, but it is adapted to the filtration of some other B.M. \(Z\), that may not be adapted to \(\mathcal{F}^S\) or \(\mathcal{F}^w\).

\(\{Y_s, s \leq t\}\) is not observable at time \(t\).

Relevant quantity to figure out is underlined not

\[
P_t(dy) = P[Y_t \in dy|\mathcal{F}^S_t]
\]

probability distribution of \(Y\) given all info up to time \(t\), because if you would have this \(Y\) would be completely observable, but rather

\[
P_t(dy) = P[Y_t \in dy|S_{\frac{t}{n}} : k = 0, 1, ..., n]
\]

implied Volatility

Here we deal again with classical B-S model: \(\sigma\) is constant.

We want to find “the market expectation” of volatility: look at all the information in the market to find some argument that volatility has some particular value.

We calculate \(\sigma\) from some well-trusted “benchmark” option in the market relative to a stock or index whose volatility you believe is equal to the one you are trying to estimate. By well-trusted we mean that we trust that the price is arbitrage-free.

Example

European call option on a stock: benchmark price = \(p = \text{call option price given strike price} u\) and current \(S(t) = S\) and maturity \(T\):

\[
p = \text{Euro call price}(u, T, S, t, r, \sigma).
\]

Since we know everything except \(\sigma\), we solve for \(\sigma\) given the market price \(p\). This is the “implied” volatility \(I(u) = \sigma\). Use this \(\sigma\) to do option pricings.

Completeness in the Black-Scholes model

Theorem

The general Black-Scholes model

\[
\frac{dS(t)}{S(t)} = \alpha(t, S(t))dt + \sigma(t, S(t))dw(t)
\]
$B(t) = e^{rt}$

is complete.

**Corollary**

To replicate $\chi = \Phi(S(T))$ with $(u^0, u^*)$ just define $F$ using B-S PDE and use

$$u^* = \frac{S \frac{\partial F}{\partial S}}{F}, \quad u^0 = \frac{F - S \frac{\partial F}{\partial S}}{F}.$$  

In practice, once BSE is solved, the only thing you really need to remember is

$$h^* = \frac{\partial F}{\partial S}$$

(i.e. BSE contains all the info about the hedging except $h^* = \frac{\partial F}{\partial S}$)

**More examples**

We looked at:  $\chi = \max(S(T) - K, 0)$-Euro call. Are

$$\chi = \max\left(\frac{1}{T} \int_0^T S(t) dt - K, 0\right)$$

$$\chi = S(T) - \inf_{0 \leq t \leq T} S(t)$$

simple options?

Asian options fall in the class of claims $\chi$ of the form $\chi = \Phi(S(T), Z(T))$ where $Z(t) = \int_0^t g(u, S(u)) du$, where $g$ is a deterministic function. In the case of Asian options $g(t, S(t)) = \frac{1}{T} S(t)$.

We need a more general theory to solve this type of claim.

Let’s define $F$ as the solution of the BSE with one additional term $g(t, S, Z) \frac{\partial F}{\partial Z}$.

**Met a theorem**

In general, in a model, if there are more traded assets (excluding the risk-free asset), then there are sources of randomness, then the market is complete.

if there are more sources of randomness than there are traded assets (excluding B), then the market is free of arbitrage (because there is more randomness than you can control).

Usually there are more sources of randomness than traded assets.