Lecture

In real life the vast majority of all traded options are written on stocks having at least one dividend left before the date of expiration of the option. Thus the study of dividends is important from a practical point of view. Also, this theory will be of use in the study of currency derivatives.

Discrete Dividends

Price Dynamics and Dividend Structure

We consider an underlying asset ("the stock") with price process $S$, over a fixed time interval $[0, T]$. For this price process at fixed points in time

$$0 < T_n < T_{n-1} < ... < T_2 < T_1 < T,$$

dividends are paid out to the holder of the stock.

The stock price structure can be described as follows:

- Between dividend points the stock price process satisfies the SDE, under the objective probability $P$

  $$dS = \alpha Sdt + \sigma Sd\bar{W},$$

  ie the $S$-process satisfies the SDE above on each half open interval of the form $[T_{i+1}, T_i), i = 1, ..., n - 1$, as well as on the intervals $[0, T_n)$ and $[T_1, T]$.

- Immediately before a dividend time $t$, i.e. at $t^- = t - dt$, we observe the stock price $S_{t^-}$.

- Given the stock price above, the size of the dividend at time $t$ is determined as $\delta(S_{t^-})$, where $\delta$ is a deterministic continuous function $\delta : R \longrightarrow R$.

Note that the assumption above guarantees that the size of the dividend $\delta$ at a dividend time $t$ is already determined at $t^-$. 

- “Between” $t^- = t - dt$ and $t$ the dividend is paid out.

- At time $t$ the stock price has a jump, determined by

  $$S_t = S_{t^-} - \delta[S_{t^-}].$$

This assumption is necessary in order to avoid arbitrage possibilities. This also means that we will view the stock price at time $t$ as the price **ex dividend**, ie after dividend

Pricing Contingent Claims

As usual we consider a fixed contingent $T$-claim of the form

$$\mathcal{X} = \Phi(S_T),$$

where $\Phi$ is some given deterministic function, and our immediate problem is to find the arbitrage free price process $\Pi(t; \mathcal{X})$ for the claim $\mathcal{X}$. We will solve this problem by a recursive procedure.
where, starting at \( T \) and then working backwards in time, we will compute \( \Pi(t; \mathcal{X}) \) for each intra-dividend interval separately.

\( T_1 \leq t \leq T \):

We start by computing \( \Pi(t; \mathcal{X}) \) for \( t \in [T_1, T] \). Since our interpretation of the stock price is ex dividend, this means that we are actually facing a problem without dividends over this interval. Thus, for \( T_1 \leq t \leq T \), we have \( \Pi(t; \mathcal{X}) = F(t, S_t) \) where \( F \) solves the usual Black-Scholes equation

\[
\begin{align*}
\frac{\partial F}{\partial t} + rs \frac{\partial F}{\partial s} + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 F}{\partial s^2} - rF &= 0 \\
F(T, s) &= \Phi(s)
\end{align*}
\]

In particular, the pricing function at \( T_1 \) is given by \( F(T_1, s) \).

\( T_2 \leq t < T_1 \):

Now we go on to compute the price of the claim for \( T_2 \leq t < T_1 \). We start by computing the pricing function \( F \) at the time immediately before \( T_1 \), i.e. at \( t = T_1^- \). Suppose therefore that we are holding one unit of the contingent claim, and let us assume that the price at time \( T_1^- \) is \( S_{T_1^-} = s \). This is the price cum dividend, and in the next infinitesimal interval the following will happen.

- The dividend \( \delta[s] \) will be paid out to the shareholders.
- At time \( T_1 \) the stock price will have dropped to \( s - \delta[s] \).
- We are now standing at time \( T_1 \), holding a contract which is worth \( F(T_1, s - \delta[s]) \). The value of \( F \) at \( T_1 \) has, however, already been computed in the previous step, so we have the \textbf{jump condition}

\[
F(T_1^-, s) = F(T_1, s - \delta[s]).
\]  

(3)

It now remains to compute \( F \) for \( T_2 \leq t < T_1 \), but this turns out to be quite easy. We are holding a contingent claim on an underlying asset which over the interval \([T_2, T_1]\) is not paying dividends. Thus the standard Black-Scholes argument applies, which means that \( F \) has to solve the usual Black-Scholes equation.

\[
\begin{align*}
\frac{\partial F}{\partial t} + rs \frac{\partial F}{\partial s} + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 F}{\partial s^2} - rF &= 0 \\
F(T_1, s - \delta[s]) &= F(T_1, s - \delta[s])
\end{align*}
\]

Over this interval. The boundary value is now given by the jump condition (3) above. Another way of putting this is to say that, over the half-open interval \([T_2, T_1]\), we have \( F(t, s) = F^1(t, s) \) where \( F^1 \) solves the following boundary value problem over the closed interval \([T_2, T_1]\).

\[
\begin{align*}
\frac{\partial F^1}{\partial t} + rs \frac{\partial F^1}{\partial s} + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 F^1}{\partial s^2} - rF^1 &= 0 \\
F^1(T_1, s - \delta[s]) &= F(T_1, s - \delta[s]).
\end{align*}
\]

Thus we have determined \( F \) on \([T_2, T_1]\).

\( T_3 \leq t < T_2 \):

Now the story repeats itself. At \( T_2 \) we will again have the jump condition

\[
F(T_2^-, s) = F(T_2, s - \delta[s]).
\]
and over the half-open interval \([T_3, T_2]\) \(F\) has to satisfy the Black-Scholes equation.

We may summarize our results, in order to stress the recursive nature of the procedure, as follows.

**Proposition**

- On the interval \([T_1, T]\) we have \(F(t, s) = F^0(t, s)\), where \(F^0\) solves the boundary value problem
  \[
  \begin{aligned}
  &\frac{\partial F^0}{\partial t} + rs \frac{\partial F^0}{\partial s} + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 F^0}{\partial s^2} - rF^0 = 0 \\
  &F^0(T, s) = \Phi(s)
  \end{aligned}
  \tag{7}
  \]

- On each half-open interval \([T_{i+1}, T_i)\) we have \(F(t, s) = F^i(t, s)\) for \(i = 1, 2, ..., \), where \(F^i\), over the closed interval \([T_{i+1}, T_i]\), solves the boundary value problem
  \[
  \begin{aligned}
  &\frac{\partial F^i}{\partial t} + rs \frac{\partial F^i}{\partial s} + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 F^i}{\partial s^2} - rF^i = 0 \\
  &F^i(T_i, s) = F^{i-1}(T_i, s - \delta[s]).
  \end{aligned}
  \tag{8}
  \]

We have assumed, so far, the standard Black-Scholes price dynamics between dividends.

We now turn to the possibility of obtaining a probabilistic “risk neutral valuation” formula for the contingent claim above, and as in the PDE approach this is done in a recursive manner.

For \(T_1 \leq t \leq T\) the situation is simple. Since we have no dividend points left we may use the old risk neutral valuation formula to obtain

\[
F^0(t, s) = e^{-r(T-t)} E_t^Q[\Phi(S_T)].
\tag{10}
\]

Here the \(Q\)-dynamics of the stock price are given by

\[
dS = rSdt + \sigma SdW,
\]

and we have used the notation \(F^0\) to emphasize that in this interval there are zero dividend points left. Note that the \(Q\)-dynamics of \(S\) above are only defined for the interval \([T_1, T]\).

For \(T_2 \leq t < T_1\) the situation is slightly more complicated. From the above Proposition we know that the pricing function, which on this interval is denoted by \(F^1\), solves the PDE

\[
\begin{aligned}
&\frac{\partial F^1}{\partial t} + rs \frac{\partial F^1}{\partial s} + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 F^1}{\partial s^2} - rF^1 = 0 \\
&F^1(T_1, s) = F^0(T_1, s - \delta[s]).
\end{aligned}
\]

We may now apply the Feynman-Kac Theorem to obtain the stochastic representation

\[
F^1(t, s) = e^{-r(T_1-t)} E_{t,s}[F^0(T_1, X_{T_1} - \delta[X_{T_1}])]
\tag{11}
\]

where the process \(X\), which at this point only acts as a computational dummy, is defined by

\[
dX = rXdt + \sigma XdW
\tag{12}
\]
Notice that the dummy process $X$ is defined by (12) over the entire closed interval $[T_2, T_1]$ (whereas the pricing function $F^1$ is the relevant pricing function only over the half-open interval $[T_2, T_1)$).

This implies that $X$ has continuous trajectories over the closed interval $[T_2, T_1]$, so in particular we see that $X_{T_1-} = X_{T_1}$. We may thus rewrite (11) as

$$F^1(t, s) = e^{-r(T_1-t)}E_{t,s}[F^0(T_1, X_{T_1-} - \delta[X_{T_1-}])],$$

still with the $X$-dynamics (12) on the closed interval $[T_2, T_1]$.

Let us now define the $Q$-dynamics for $S$ over the closed interval $[T_2, T_1]$ by writing

$$dS = rSdt + \sigma SdW,$$

for the half-open interval $[T_2, T_1)$ and adding the jump condition

$$S_{T_1} = S_{T_1-} - \delta[S_{T_1-}].$$

In this notation we can now write (13) as

$$F^1(t, s) = e^{-r(T_1-t)}E_{t,s}^Q[F^0(T_1, S_{T_1})],$$

and plugging in our old expression for $F^0$, we obtain

$$F^1(t, s) = e^{-r(T_1-t)}E_{t,s}^Q[e^{-r(T-T_1)}E_{T_1,S(T_1)}^Q[\Phi(S_T)]]].$$

Taking the discount factor out of the expectation, and using standard rules for iterated conditional expectations, this formula can be reduced to

$$F^1(t, s) = e^{-r(T-T_1)}E_{t,s}^Q[\Phi(S_T)].$$

We may now iterate this procedure for each intra-dividend interval to obtain the following risk neutral valuation result, which we formulate in its more general version.

**Proposition 5 (Risk neutral valuation)**

Consider a $T$-claim of the form $\Phi(S_T)$ as above. Assume that the price dynamics between dividends are given by

$$dS_t = \alpha(t, S_t)S_tdt + \sigma(t, S_t)S_t\bar{d}W,$$

and that the dividend size at a dividend point $t$ is given by

$$\delta = \delta[S_t-].$$

Then the arbitrage free pricing function $F(t, s)$ has the representation

$$F(t, s) = e^{-r(T-t)}E_{t,s}^Q[\Phi(S_T)],$$

where the $Q$-dynamics of $S$ between dividends are given by

$$dS_t = rS_tdt + \sigma(t, S_t)S_tW_t,$$

with the jump condition

$$S_t = S_{t-} - \delta[S_{t-}]$$

(14)
at each dividend point, i.e. at $t = T_1, T_2, ..., T_n$.

We end this section by specializing to the case when we have the standard Black-Scholes dynamics
\begin{equation}
    dS = \alpha S dt + \sigma S d\bar{W}
\end{equation}

between dividends, and the dividend structure has the particularly simple form
\begin{equation}
    \delta[s] = s\delta,
\end{equation}

where $\delta$ on the right-hand side denotes a positive constant.

As usual we consider the $T$-claim $\Phi(S_T)$ and, in order to emphasize the role of the parameter $\delta$, we let $F_\delta(t, s)$ denote the pricing function for the claim $\Phi$. In particular we observe that $F_0$ is our standard pricing function for $\Phi$ in a model with no dividends at all. Using the risk neutral valuation formula above, it is not hard to prove the following result.

**Proposition**

Assume that the $P$-dynamics of the stock price and the dividend structure are given by (17)-(18). Then the following relation holds.
\begin{equation}
    F_\delta(t, s) = F_0(t, (1 - \delta)^n \cdot s),
\end{equation}

where $n$ is the number of dividend points in the interval $(t, T]$.

The point of this result is of course that in the simple setting of (17)-(18) we may use our “old” formulas for no dividend models in order to price contingent claims in the presence of dividends. In particular we may use the standard Black-Scholes formula for European call options in order to price call options on a dividend paying stock. Note, however, that in order to obtain these nice results we must assume both (17) and (18).

**Continuous Dividends**

We consider the case when dividends are paid out continuously in time. As usual $S_t$ denotes the price of the stock at time $t$, and by $D(t)$ we denote the cumulative dividends over the interval $[0, t]$. Put in differential form this means that over the infinitesimal interval $(t, t + dt]$ the holder of the stock receives the amount $dD(t) = D(t + dt) - D(t)$.

**Continuous Dividend Yield**

We start by analyzing the simplest case of continuous dividends, which is when we have a continuous dividend yield.

**Assumption**

The price dynamics, under the objective probability measure, are given by
\begin{equation}
    dS_t = S_t \cdot \alpha(S_t) dt + S_t \cdot \sigma(S_t) d\bar{W}.
\end{equation}

the dividend structure is assumed to be of the form
\begin{equation}
    dD(t) = S_t \cdot \delta[S_t] dt,
\end{equation}

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where $\delta$ is a continuous deterministic function.

The most common special case is of course when the function $\alpha$ and $\sigma$ above are deterministic constants, and when the function $\delta$ is a deterministic constant. We note that, since we have no discrete dividends, we do not have to worry about the interpretation of the stock price as being ex dividend or cum dividend.

The problem to be solved is again that of determining the arbitrage free price for a $T$-claim of the form $\Phi(S_T)$. This turns out to be quite easy, and we can in fact follow the strategy. More precisely we recall the following scheme:

1. Assume that the pricing function is of the form $F(t, S_t)$.
2. Consider $\alpha, \sigma, \Phi, F, \delta$ and $r$ as exogenously given.
3. Use the general results to describe the dynamics of the value of a hypothetical self-financed portfolio based on the derivative instrument and the underlying stock.
4. Form a self-financed portfolio whose value process $V$ has a stochastic differential without any driving Wiener process, i.e. it is of the form
   \[ dV(t) = V(t)k(t)dt. \]
5. Since we have assumed absence of arbitrage we must have $k = r$.
6. The condition $k = r$ will in fact have the form of a partial differential equation with $F$ as the unknown function. In order for the market to be efficient $F$ must thus solve this PDE.
7. The equation has a unique solution, thus giving us the unique pricing formula for the derivative, which is consistent with absence of arbitrage.

We now carry out this scheme and we get:

**Proposition**
The pricing function $F(t, s)$ of the claim $\Phi(S_T)$ solves the boundary value problem

\[
\begin{align*}
\frac{\partial F}{\partial t} + \left( r - \delta \right) s \frac{\partial F}{\partial s} + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 F}{\partial s^2} - r F &= 0 \\
F(T, s) &= \Phi(s). 
\end{align*}
\] (22)

Applying the Feynman-Kac representation theorem immediately gives us a risk neutral valuation formula.

**Proposition (Pricing equation)**
The pricing function has the representation

\[ F(t, s) = e^{-r(T-t)} E^Q_{t,s}[\Phi(S_T)], \] (23)

where the $Q$-dynamics of $S$ are given by

\[ dS_t = (r - \delta[S_t])S_t dt + \sigma(S_t)S_t dW_t, \] (24)

In contrast with the case of discrete dividends we see that the appropriate martingale measure in the dividend case differs from that of the no dividend case.
Proposition (Risk neutral valuation)
Under the martingale measure \( Q \), the normalized gain process
\[
G^Z(t) = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_\tau} dD(\tau),
\]
is a \( Q \)-martingale.

Note that this property is quite reasonable from an economic point of view: in a risk neutral world today’s stock price should be the expected value of all future discounted earnings which arise from holding the stock. In other words, we expect that
\[
S(0) = E^Q\left[ \int_0^t e^{-r\tau} dD(\tau) + e^{-rt}S(t) \right],
\]
and in the exercise the reader is invited to prove this “cost of carry” formula.

As in the discrete case it is natural to analyze the pricing formulas for the special case when we have the standard Black-Scholes dynamics
\[
dS = \alpha S dt + \sigma S d\bar{W},
\]
where \( \alpha \) and \( \sigma \) are constants. We also assume that the dividend function \( \delta \) is a deterministic constant. This implies that the martingale dynamics are given by
\[
dS = (r - \delta) S dt + \sigma S d\bar{W},
\]
i.e. \( S \) is geometric Brownian motion also under the risk adjusted probabilities. Again we denote the pricing function by \( F_\delta \) in order to highlight the dependence upon the parameter \( \delta \). It is now easy to prove the following result, which shows how to price derivatives for a dividend paying stock in terms of pricing functions for a nondividend case.

Proposition
Assume that the functions \( \sigma \) and \( \delta \) are constant. Then, with notation as above, we have
\[
F_\delta(t, s) = F_0(t, se^{-\delta(T-t)}).
\]

The general case. This would be a good subject for a project.

We now consider a more general dividend structure, which we will need when dealing with futures contracts.

Assumption
The price dynamics, under the objective probability measure, are given by
\[
dS_t = S_t \cdot \alpha(S_t) dt + S_t \cdot \sigma(S_t) d\bar{W}_t,
\]
The dividend structure is assumed to be of the form
\[
dD(t) = S_t \cdot \delta[S_t] dt + S_t \gamma[S_t] d\bar{W}_t,
\]
where $\delta$ and $\gamma$ are continuous deterministic functions.

We again consider the pricing problem for a contingent $T$-claim of the form $X = \Phi(S_T)$, and the only difference from the continuous yield case is that now we have to assume that the pricing function for the claim is a function of $D$ as well as $S$. We thus assume a claim price process of the form

$$\Pi(t; X) = F(t, S_t, D_t),$$

and then we carry out the standard program 1-7 of the previous section.

After a large number of simple, but messy and extremely boring, calculations which (needless to say) are left as an exercise, we end up with the following result.

**Proposition (Pricing equation)**

The pricing function $F(t, s, D)$ for the claim $X = \Phi(S_T)$ solves the boundary value problem

$$\begin{cases}
\frac{\partial F}{\partial t} + AF - rF = 0 \\
F(T, s, D) = \Phi(s)
\end{cases},$$

(31)

where

$$AF = \left(\frac{\alpha \gamma + \sigma r - \delta \sigma}{\sigma + \gamma}\right)s \frac{\partial F}{\partial s} + \left(\frac{\delta \sigma + \gamma r - \gamma \alpha}{\sigma + \gamma}\right)s \frac{\partial F}{\partial D} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} + \frac{1}{2} \gamma^2 s^2 \frac{\partial^2 F}{\partial D^2} + \sigma \gamma s \frac{\partial^2 F}{\partial s \partial D}.$$

Using the Feynman-Kac technique we have the following risk neutral valuation result.

**Proposition (Risk neutral valuation)**

The pricing function has the representation

$$F(t, s, D) = e^{-r(T-t)} E^Q_t, s, D[\Phi(S_T)],$$

where the $Q$-dynamics of $S$ and $D$ are given by

$$dS_t = S_t \left(\frac{\alpha \gamma + \sigma r - \delta \sigma}{\sigma + \gamma}\right)dt + S_t \sigma dW_t,$$

$$dD_t = S_t \left(\frac{\delta \sigma + \gamma r - \gamma \alpha}{\sigma + \gamma}\right)dt + S_t \gamma dW_t.$$

**Remark**

In the expressions of the propositions above we have suppressed $S_t$ and $s$ in the functions $\alpha, \sigma, \delta, \gamma$.

The role of the martingale measure is the same as in the previous section.

**Proposition**

The martingale measure $Q$ is characterized by the following facts:

- There exists a market price of risk process $\lambda$ such that the $Q$-dynamics are in the form

$$dS = S(\alpha - \lambda \sigma)dt + S\sigma dW,$$

$$dD = S(\delta - \lambda \gamma)dt + S\gamma dW.$$
• the normalized gains process \( G_Z \), defined by

\[
G^Z(t) = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_\tau} dD(\tau),
\]

is a \( Q \)-martingale.

This result has extensions to multidimensional factor models. We will not go into details, but are content with stating the main result.

**Proposition**
Consider a general factor model. If the market is free of arbitrage, then there will exist universal market price of risk process \( \lambda = (\lambda_1, ..., \lambda_k)^* \) such that

• For any \( T \)-claim \( \mathcal{X} \) the pricing function \( F \) has the representation

\[
F(t, x) = E_{t,x}^Q[e^{-\int_t^T r(X(u))du} \cdot \Phi(X(T))].
\]  

(32)

• The \( Q \)-dynamics of the factor processes \( X_1, ..., X_k \) are of the form

\[
dX_i = (\mu_i - \delta_i \lambda) dt + \delta_i dW, \quad i = 1, 2, ..., k.
\]

• For any price process \( S \) (underlying or derivative) with dividend process \( D \), the normalized gains process

\[
Z_t = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_\tau} dD(\tau),
\]

is a \( Q \)-martingale.