First fundamental theorem of finance

Proof: Let \( W = \{ G^x \in \mathbb{R}^k \mid \text{for the set of all possible discounted gain processes for some trading strategy} \} \)

Remark: \( W \) is a set of random variables

6) If \( x \in W \) then \( x \) is a time 1 discounted wealth when the initial investment is 0.

c) \( W \) is a linear subspace of \( \mathbb{R}^k \) (if \( x, y \in W \) then \( x + y \in W \))

Consider the set \( A = \{ x \in \mathbb{R}^k \mid x \geq 0, x \neq 0 \} \)

By the def. of arbitrage, there is arbitrage

iff \( W \cap A \neq \emptyset \)

So if we assume there is no risk probability measure we will have to find something in \( W \cap A \neq \emptyset \).

Denote by \( W^\perp = \{ y \in \mathbb{R}^k \mid x \cdot y = 0 \text{ for all } x \in W \} \)

the orthogonal subspace.

\( p^+ \) is the line \( x+y=1 \)
(in $\mathbb{R}^2$ the only linear subspace on lines and axes. Hence
$W$ is a line $\Rightarrow$ $W^{-1}$ is the orthogonal line on $W$) From
the picture we could see $W^{-1} \cap \mathcal{A} = \emptyset$.
Hence there is a point in $W^{-1} \cap \mathcal{A}$ that has all co-
ordinates nicely positive and such that they add up to 1, so they can be
interpreted as a probability measure.

In other words, denote
\[ \mathcal{P}^+ = \{ x \in \mathbb{R}^k : x_1 + \ldots + x_k = 1, x_1 > 0, \ldots, x_k > 0 \} \]
The geometry suggests that $W \cap \mathcal{A} = \emptyset \iff W^{-1} \cap \mathcal{P}^+ = \emptyset$.

Since $\Delta S_m \in W$ for all $m$, it follows that any element
of the set $W^+ \cap \mathcal{P}^+$ is actually a risk neutral probability
measure (it is a probability on such that
\[ \sum \Delta S_m \langle w_k \rangle x_k = 0 \])

Let's prove the converse now: assume we have a risk neutral
probability measure $\mathcal{Q}$ and show that there is no
arbitrage.

If $\mathcal{Q}$ is a risk neutral probability measure, then for any
$G \in W$ we have
\[ E_\mathcal{Q} G = E_\mathcal{Q} \left( \sum_{m=1}^{\infty} \Delta S_m \langle w_k \rangle \right) = \sum_{m=1}^{\infty} E_\mathcal{Q} \Delta S_m = 0 \]
so $\mathcal{Q} \in W^+ \cap \mathcal{P}^+$. Thus, if we let $\mathcal{M}$ denote the set of
all risk neutral probabilities we have
\[ \mathcal{M} = W^+ \cap \mathcal{P}^+ \]

By the above picture, we have $W \cap \mathcal{A} = \emptyset$ iff $\mathcal{M} = \emptyset$. 

\[ x_1, \ldots, x_k \]