European arithmetic Asian call option pays the difference, if positive, between the arithmetic average of the asset price
\[ A_T = \frac{1}{N} \sum_{i=1}^{N} S_{t_i} \]
and the strike price \( K \) at the maturity date \( T \). The arithmetic average is taken on a set of observations (fixing) of the asset price \( S_{t_i} \), which we assume to follow GBM, at dates \( t_1, ..., t_N \). Thus the pay-off at the maturity date is
\[ \max (A_T - K, 0). \]

There is no analytical formula for the price of an arithmetic Asian option. However, there is a simple analytical formula for the price of a geometric Asian option. A geometric Asian call option pays the difference, if positive, between the geometric average of the asset price
\[ G_T = \left( \prod_{i=1}^{N} S_{t_i} \right)^{1/N} \]
and the strike price \( K \) at the maturity date \( T \). Since the geometric average is essentially the product of lognormally distributed variables then it is also lognormally distributed. Therefore, the price of the geometric Asian call option is given by a modified Black-Scholes formula (obtaining such formula using risk neutral valuation is a good project for one person):
\[
C_{\text{Geometric Asian}} = \exp (-rT) \left[ \exp \left( a + \frac{b}{2} \right) N(x) - KN \left( x - \sqrt{b} \right) \right]
\]
where
\[
a = \frac{m}{N} \ln (G_t) + \frac{N - m}{N} \left[ \ln (S) + \nu (t_{m+1} - t) + \frac{\nu}{2} (T - t_{m+1}) \right]
\]
\[
b = \frac{(N - m)^2}{N^2} \sigma^2 (t_{m+1} - t) + \frac{\sigma^2 (T - t_{m+1})}{6N^2} (N - m) (2 (N - m) - 1)
\]
\[
\nu = r - \delta - \frac{\sigma^2}{2}
\]
\[
x = \frac{a - \ln (K) + b}{\sqrt{b}}.
\]
Here \( G_t \) is the current geometric average and \( m \) is the last known observation (fixing). The geometric Asian option makes a good static hedge style control variate for the arithmetic Asian.

The following pseudo-code is an implementation of the Monte Carlo valuation of a European Asian call option with a geometric Asian call option control variate. We simulate the difference between the arithmetic and geometric Asian options, i.e. a hedged portfolio which is long one arithmetic Asian and short one geometric Asian option. This is much faster than using the delta of the geometric Asian option to generate a delta hedge control variate because we do not have to compute the delta at every time step and it is equivalent to a continuous delta hedge. Note the bold highlighted lines in the pseudo-code that precompute the drift and volatility constant expressions required for the simulation of the asset price between the observation (fixing) dates. This increases the efficiency of the simulation significantly because these contents are used for every time step in every simulation.
Exercise 1  Price a one-year maturity, European Asian call option with a strike price of 100, current asset price at 100 and volatility of 20%. The continuously compounded interest rate is assumed to be 6% per annum, the asset pays a continuous dividend yield of 3% per annum, and there are 10 equally spaced observation (fixing) dates. Simulate 10 time steps and 100 simulations. Thus

\[ K = 100 \]
\[ T = 1 \text{ year} \]
\[ S = 100 \]
\[ \sigma = 0.2 \]
\[ r = 0.06 \]
\[ \delta = 0.03 \]
\[ N = 10 \]
\[ M = 100 \]