Hence, if we generate a path using as inputs $U$ we expect a method of antithetic variates to reduce variance. Therefore, $\hat{\sqrt{Y}}$ is the sample mean of the $n$ independent observations

$$\left( \frac{Y_1 + \tilde{Y}_1}{2} \right), \left( \frac{Y_2 + \tilde{Y}_2}{2} \right), ..., \left( \frac{Y_n + \tilde{Y}_n}{2} \right).$$

By Central Limit theorem,

$$\frac{\sqrt{n} \left( \hat{Y}_{AV} - E[Y] \right)}{\sigma_{AV}}$$

converges to $N(0,1)$ in distribution, where

$$\sigma_{AV}^2 = Var \left( \frac{Y_i + \tilde{Y}_i}{2} \right)$$

$$= \frac{1}{4} \left( Var [Y_i] + Var [\tilde{Y}_i] + 2Cov [Y_i, \tilde{Y}_i] \right)$$

$$= \frac{1}{4} \left( 2Var [Y_i] + 2Cov [Y_i, \tilde{Y}_i] \right)$$

$$= \frac{1}{2} \left( Var [Y_i] + Cov [Y_i, \tilde{Y}_i] \right).$$

Thus, we need $Cov [Y_i, \tilde{Y}_i] < 0$ for the pairs $Y_i$ and $\tilde{Y}_i$ to reduce variance.
Implementation of this method to options pricing is very simple. For example, consider pricing a European call option. Our simulated payoffs are

\[ C_{T,j} = \max \left( 0, S e^{\nu T + \sigma \sqrt{T} \epsilon_j} - K \right). \]

We can simulate the payoffs to the option on the perfectly negatively correlated asset as

\[ \overline{C}_{T,j} = \max \left( 0, S e^{\nu T - \sigma \sqrt{T} \epsilon_j} - K \right) \]

where \( \epsilon_j \sim N(0,1) \). In other words, we simply replace \( \epsilon_j \) by \( -\epsilon_j \) in the equation (1) for the simulation. We then take the average of the two payoffs as the payoff for that simulation.

Note that, not only do we obtain a much more accurate estimate from \( M \) pairs of \( (C_{T,j}, \overline{C}_{T,j}) \) than from \( 2M \) of \( C_{T,j} \), but it is also computationally cheaper to generate the pair \( (C_{T,j}, \overline{C}_{T,j}) \) than two instances of \( C_{T,j} \).

**Exercise 3** By using antithetic variates method, price a one-year maturity, \( T = 1 \), at-the-money, \( K = S_0 \), European call option with the current asset price \( S_0 = 100 \) and the volatility \( \sigma = 20\% \). The continuously compounded interest rate is assumed to be \( r = 6\% \) per annum. The asset pays a continuous dividend \( \delta = 3\% \) per annum. Simulate the price with \( N = 10 \) time steps and \( M = 10000 \) simulations. Calculate the standard error of the estimated price. Compare and comment on the standard error of estimated price with the previous (identical) exercise that does not use variance reduction.