Hedge Sensitivities-The Greeks

Importance of Greeks

The fundamental key to option pricing is to calculate the cost of replication of the sold derivative contract. For some options, we can construct a static replication strategy. The price that we need to charge is then simply given by the cost of setting up the initial hedge. Having done that, we may forget about this position, since we are hedged. Some people call this hedging strategy as hedge-and-forget.

For most options, though, there is no static replication, and we need to use dynamic hedging strategy in order to protect us from the market risk posed by our short position in the option. In some sense, the quantitative analyst’s job is done once a stochastic process for the underlying asset price is chosen, the risk-neutral measure is established, and the value of the derivative is determined. However, for the trader who has to manage the short position of the derivative, the fun is only just started. The position now has to be rehedged dynamically, which requires knowledge of the various hedge parameters known as the Greeks, owing to the market practice of using the Greek letters to represent risk parameters. This means, we are not only need to be able to value the option, but also to calculate how its value depends on changes in model parameters and the traded price of the underlying asset.

Defining Greeks

Let $C(t, S, \sigma, r)$ be the price of an option at the current time $t$ with underlying asset price $S$, volatility $\sigma$ and risk-free interest rate $r$. The delta $\Delta$ of an option is the rate of change of the option price with respect to the underlying asset price

$$\Delta = \frac{\partial C}{\partial S}.$$  

Delta hedging involves creating a position, called delta-neutral position, with zero delta. Because the delta of the underlying asset is 1.0, one way of hedging is to take a position of $-\Delta$ in the underlying asset for each long option being hedged. The delta of an option changes over time. This means that the position in the underlying asset has to be frequently adjusted.

Once an option position has been made delta neutral, the next stage is often to look at its gamma $\Gamma$. The gamma $\Gamma$ of an option is the rate of change of its delta with respect to the price of the underlying asset

$$\Gamma = \frac{\partial^2 C}{\partial S^2}.$$  

It is a measure of the curvature of the relationship between the option price and the asset price. The impact of this curvature on the performance of delta hedging can be reduced by making an option position gamma neutral. If $\Gamma$ is the gamma of the position being hedged, the reduction is usually achieved by taking a position in a traded option that has a gamma of $-\Gamma$.

Delta and gamma hedging are both based on the assumption that the volatility of the underlying asset is constant. In practice, volatilities do change over time. The vega of an option measures the rate of change of option value with respect to volatility

$$v\text{ega} = \frac{\partial C}{\partial \sigma}.$$  

A trader who wishes to hedge an option position against volatility changes can make the position vega neutral. As with the procedure for creating gamma neutrality, this usually involves taking an offsetting position in a traded option. If the trader wishes to achieve both gamma and vega neutrality, two traded options are usually required.

Two other measures of the risk of an option position are theta $\Theta$ and rho $\rho$. Theta measures the rate of change of the value of the option value with respect to the passage of time

$$\Theta = \frac{\partial C}{\partial t}.$$  

and rho measures the rate of change of the option value with respect to the interest rate

\[ \rho = \frac{\partial C}{\partial r} \]

with all else remaining constant.

In practice, option traders usually rebalance their portfolios at least once a day to maintain delta neutrality. It is usually not feasible to maintain gamma and vega neutrality on a regular basis. Typically a trader monitors these measures. If they get too large, either corrective action is taken or trading is curtailed.

**Computing Greeks**

We may approximate the greeks by finite difference ratios.

\[
\Delta \approx \frac{C(t, S + \delta, \sigma, r) - C(t, S - \delta, \sigma, r)}{2\delta} \\
\Gamma \approx \frac{C(t, S + \delta, \sigma, r) - 2C(t, S, \sigma, r) + C(t, S - \delta, \sigma, r)}{\delta^2} \\
\text{vega} \approx \frac{C(t, S, \sigma + \delta, r) - 2C(t, S, \sigma - \delta, r)}{2\delta} \\
\Theta \approx \frac{C(t + \delta, S, \sigma, r) - C(t, S, \sigma, r)}{\delta} \\
\rho \approx \frac{C(t, S, \sigma, r + \delta) - C(t, S, \sigma, r - \delta)}{2\delta}
\]

where \( C(t, S + \delta, \sigma, r) \) is the Monte Carlo estimate using an initial asset price of \( S + \delta \) and \( \delta > 0 \). Other \( C \)'s are defined similarly. Note that every price \( C \) above should be computed using the same set of random numbers. If this is not done then the random error in the prices from the Monte Carlo simulation can be a large proportion of the price differences in the numerator of the finite difference ratios leading to very large errors in the sensitivity estimates. By using the same random numbers, the pricing errors will tend to cancel out.

A more efficient way to compute delta is by applying the discounted expectation approach. We can express the standard European call delta as follows.

\[
\Delta = \frac{\partial C}{\partial S} = \frac{\partial}{\partial S} \left( e^{-rT} E \left[ (S_T - K) 1_{S_T > K} \right] \right) \tag{1}
\]

where

\[ S_T = S e^{\nu T + \sigma Z_T} \]

Substituting \( S_T \) into equation (1) and differentiating, we obtain

\[
\Delta = e^{-rT} E \left[ e^{\nu T + \sigma Z_T} 1_{S_T > K} \right] \tag{2}
\]

Hence, equation (2) is our basis for the Monte Carlo simulation of \( \Delta \). You may also apply the same technique for vega, theta and rho. This technique cannot be used for the calculation of gamma because differentiating equation (2) again leads to the expectation of a Dirac delta function which cannot easily be evaluated by Monte Carlo simulation. We can, however, use a finite difference ratio in terms of delta.

**Exercise 1** Compute the delta of a one year maturity, at-the-money European call option with the current asset price at $100 and volatility 20%. The continuously compounded interest rate is assumed to be 6% per annum and the asset pays a continuous dividend yield of 3% per annum. Use 100 simulations and one time step.

The antithetic variate methods can be applied in the same way as we had discussed before.