The method of control variates is among the most effective and broadly applicable technique for improving the efficiency of Monte Carlo simulation. It exploits information about errors in estimates of known quantities to reduce the error in an estimate on an unknown quantity.

Here is how it works:

Let $Y_1, Y_2, \ldots, Y_n$ be outputs from $n$ replications of a simulation. For example $Y_i$ could be the discounted payoff of a derivative security on the $i$th simulated path. Suppose that the $Y_i$ are iid and that our objective is to estimate $E(Y_i)$. The usual estimator that we used so far is the sample mean $\bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i$. This estimator is unbiased and converges with probability 1 as $n \to \infty$. Suppose that on each replication we calculate another output $X_i$ along with $Y_i$. Suppose that the pairs $(X_i, Y_i)$ are iid and that the expectation of $X_i$, $E(X_i)$ is known. Then for any $b$ fixed we can compute

$$Y_i(b) = Y_i - b(X_i - E(X))$$

and its sample mean

$$\bar{Y}(b) = \frac{1}{n} \sum_{i=1}^{n} (Y_i - b(X_i - E(X)))$$

This is called a control variate estimator. The observed error $\tilde{X} = E(X)$ serves as a control in estimating $E(Y)$.

As an estimator of $E(Y)$ the control variate estimator is unbiased and consistent.

Now observe that:

If $Var(Y_i(b)) = \sigma^2(b)$, then $Var(\bar{Y}(b)) = \frac{\sigma^2(b)}{n}$.

$Var(\bar{Y}) = \sigma^2_X/n$.

Therefore the control variate estimator has smaller variance than the standard estimator if $b^2 \sigma_x < 2b \sigma_y \rho_{XY}$. Therefore the optimal coefficient $b^*$ that minimizes the variance is

$$b^* = \frac{\sigma_y \rho_{XY}}{\sigma_x} = \frac{Cov(X,Y)}{Var(X)}$$

and the ratio

$$\frac{Var(\bar{Y}(b^*))}{Var(\bar{Y})} = 1 - \rho^2_{XY}$$

Consider now the case of writing a European call option. If we try to estimate the call value as the mean of a number of Monte-Carlo simulations then the standard deviation of the mean will be large.

Now, consider delta hedging, which is a dynamic rebalancing portfolio strategy that replicate the payoff of the call option. The pay-off of the hedged portfolio has a much smaller standard deviation. Let us consider the mechanics of a discretely rebalanced delta hedge in detail. The delta hedge consists of holding $\frac{\partial C}{\partial S}$ number of the underlying asset and is balanced at each time $t_0, t_1, \ldots, t_N = T$ where $t_i - t_{i-1} = \Delta t$. The hedging procedure consists of selling the option, putting the option premium in the bank and rebalancing the holding in the asset at each time $t_i$ which results to cash flows into and out of the bank account. At the maturity date $T$ of the hedge, the portfolio consists of the bank account and the asset whose value closely replicate the pay-off of the option. This strategy can be expressed as follows.

$$C_{t_0} e^{r(T-t_0)} - \sum_{i=0}^{N} \left( \frac{\partial C_t}{\partial S} - \frac{\partial C_{t+1}}{\partial S} \right) S_{t_i} e^{r(T-t_i)} \right) = C_T + \eta$$

(1)
where

\[ \frac{\partial C_{t_{i-1}}}{\partial S} = 0 \]

\( C_{t_0} e^{r(T-t_0)} \) = Option premium at maturity date

\( \left( \frac{\partial C_{t_i}}{\partial S} - \frac{\partial C_{t_{i-1}}}{\partial S} \right) S_t e^{r(T-t_i)} \) = Cash flows from rebalancing the hedge at time \( t_i \)

\( C_T = \max \left( S_T - K, 0 \right) \), the pay-off of the option

\( \eta = \text{Hedging error} \)

By letting \( \frac{\partial C_{t_N}}{\partial S} \) \( S_{t_N} = 0 \), which corresponds to liquidate the holding of the asset at time \( t_{N-1} \) into cash, and grouping all terms with \( \frac{\partial C_{t_i}}{\partial S} \) at the same time step, we may write Equation (1) as

\[ C_{t_0} e^{r(T-t_0)} + \left[ \sum_{i=0}^{N-1} \frac{\partial C_{t_i}}{\partial S} \left( S_{t_{i+1}} - S_{t_i} e^{r \Delta t_i} \right) e^{r(T-t_{i+1})} \right] = C_T + \eta. \]

Note that \( S_{t_i} e^{r \Delta t_i} = E \left[ S_{t_{i+1}} \right] \) whose expectation is over the risk-neutral probability measure. We define a delta-based control variate as

\[ cv_1 = \sum_{i=0}^{N-1} \frac{\partial C_{t_i}}{\partial S} \left( S_{t_{i+1}} - E \left[ S_{t_{i+1}} \right] \right) e^{r(T-t_{i+1})} \]

whose mean \( E [cv_1] = 0 \). Hence the initial option premium is

\[ C_{t_0} = e^{-r(T-t_0)} \left( C_T - cv_1 \right) + \eta_1 \] (2)

where the hedge error \( \eta_1 = e^{-r(T-t_0)} \eta \) is negligible for reasonably frequent in rebalancing the portfolio, i.e. by setting \( \Delta t \) small.

**Exercise 1** Price a one-year maturity, at-the-money European call option by Monte Carlo simulation with a Delta-based control variate for a total of 100 simulations with 10 time steps. The current asset price is $100 and volatility is 20%. The continuously compounded interest rate is assumed to be 6% per annum and the asset pays a continuous dividend yield of 3% per annum.

It is straightforward to combine the antithetic and control variate methods. We simply accumulate control variates for the standard and antithetic asset paths.

**Exercise 2** Price the option in previous exercise by Monte Carlo simulation with antithetic and Delta-based control variates.

In the same way as for deriving delta-based control variate, we can construct other control variates equivalent to their hedges. For example, a control variate for gamma hedge is

\[ cv_2 = \sum_{i=0}^{N-1} \frac{\partial^2 C_{t_i}}{\partial S^2} \left( (\Delta S_{t_i})^2 - E \left[ (\Delta S_{t_i})^2 \right] \right) e^{r(T-t_{i+1})} \]

where \( E \left[ (\Delta S_{t_i})^2 \right] = S_{t_i}^2 \left( e^{(2\sigma^2+\sigma^2) \Delta t_i} - 2e^{\sigma \Delta t_i} + 1 \right) \).

For the general case of a European option paying off \( C_T \) at time \( T \), Equation (2) becomes

\[ C_{t_0} = e^{-r(T-t_0)} \left( C_T - \sum_{k=1}^{m} \beta_k cv_k \right) + \eta \]

where the \( \beta \) factors are included to account for the sign of the hedge.

**Exercise 3** Price the option in previous exercise by Monte Carlo simulation with antithetic, Delta and Gamma-based control variates.