MATH 60093 Monte Carlo Modeling

Multiple Stochastic Factors

One of the main uses of Monte Carlo simulation is for pricing options under multiple stochastic factors. Consider pricing options whose payoff depends on multiple asset prices. For example, a European spread option on the two asset prices $S_1$ and $S_2$ has payoff given by

$$\max (S_1 - S_2 - K, 0).$$

(1)

Assume that $S_1$ and $S_2$ follow GBM

$$dS_1 = (r - \delta_1) S_1 dt + \sigma_1 S_1 dW_1$$
$$dS_2 = (r - \delta_2) S_2 dt + \sigma_2 S_2 dW_2$$

where the Brownian motions $W_1$ and $W_2$ have instantaneous correlation $\rho$, that is the covariance $\text{cov}(dW_1, dW_2) = \rho dt$. In order to price the option by simulation, we use the solution of the SDEs to simulate the asset prices.

$$S_1(T) = S_1(0) \exp \left( \nu_1 T + \sigma_1 \sqrt{T} z_1 \right)$$
$$S_2(T) = S_2(0) \exp \left( \nu_2 T + \sigma_2 \sqrt{T} z_2 \right)$$

where

$$\nu_1 = r - \delta_1 - \frac{1}{2} \sigma_1^2$$
$$\nu_2 = r - \delta_2 - \frac{1}{2} \sigma_2^2$$

and $z_1$ and $z_2$ are random variables generated from the standard bivariate normal distribution with correlation $\rho$. The correlated random variables can be easily generated from two independent standard normal random variables $\epsilon_1$ and $\epsilon_2$ and combining them as follows:

$$z_1 = \epsilon_1$$
$$z_2 = \rho \epsilon_1 + \sqrt{1 - \rho^2} \epsilon_2$$

The Monte Carlo procedure is exactly the same as that for the standard European call except that we simulate the two asset price processes evaluate the payoff (1).

Exercise 1 Price a one-year maturity, European call option with strike price $K = 1$ whose current asset prices are $S_1(0) = 100$ and $S_2(0) = 110$, volatilities $\sigma_1 = 20\%$ and $\sigma_2 = 30\%$, continuously compounded interest rate is 6% per annum and continuously compounded dividends are $\delta_1 = 3\%$ and $\delta_2 = 4\%$.

In the same way, if we want to price an option under more general stochastic processes such as stochastic volatility and/or stochastic interest rates, we simply simulate the required stochastic processes.

Exercise 2 Price a one-year maturity, European call option with strike price $K = 1$ whose current asset prices are $S_1(0) = 100$ and $S_2(0) = 110$, current volatilities $\sigma_1(0) = 20\%$ and $\sigma_2(0) = 30\%$, continuously compounded interest rate is 6% per annum and continuously compounded dividends are $\delta_1 = 3\%$ and $\delta_2 = 4\%$. Assume the asset prices $S_1$ and $S_2$ follows the following stochastic processes

$$dS_1 = (r - \delta_1) S_1 dt + \sigma_1 S_1 dW_1$$
$$dS_2 = (r - \delta_2) S_2 dt + \sigma_2 S_2 dW_2$$

(2)
where the volatilities $\sigma_1$ and $\sigma_2$ are stochastic and follow the following stochastic processes for variance of its returns $V_1$ and $V_2$

\[
\begin{align*}
    dV_1 &= \alpha_1 (\overline{V}_1 - V_1) \, dt + \xi_1 \sqrt{V_1} \, dW_3 \\
    dV_2 &= \alpha_2 (\overline{V}_2 - V_2) \, dt + \xi_2 \sqrt{V_2} \, dW_4
\end{align*}
\]  

(3)

with $V_i = \sigma_i^2$, $\alpha_i$ is the rate of mean reversion on the variance process $V_i$, $\overline{V}_i$ is the long term mean of $V_i$, $\xi_i$ is the volatility of the variance $V_i$. Now, suppose

\[
\begin{align*}
    \alpha_1 &= 1.0 \\
    \alpha_2 &= 2.0 \\
    \overline{V}_1 &= 0.04 \\
    \overline{V}_2 &= 0.09 \\
    \xi_1 &= 0.05 \\
    \xi_2 &= 0.06
\end{align*}
\]

and the Brownian motions $W_i$ have the following correlation matrix

\[
\rho = \begin{pmatrix}
    1 & \rho_{12} & \rho_{13} & \rho_{14} \\
    \rho_{12} & 1 & \rho_{23} & \rho_{24} \\
    \rho_{13} & \rho_{23} & 1 & \rho_{34} \\
    \rho_{14} & \rho_{24} & \rho_{34} & 1
\end{pmatrix} = \begin{pmatrix}
    1 & 0.50 & 0.20 & 0.01 \\
    0.50 & 1 & 0.01 & 0.30 \\
    0.20 & 0.01 & 1 & 0.30 \\
    0.01 & 0.30 & 0.30 & 1
\end{pmatrix}
\]

In this case, we need to generate four correlated normal random variables in order to simulate the four processes in (2) and (3).