Midterm Review : Analysis I - Spring 2019 – Dr. Smithies. Our midterm is Wed Oct 23rd.

1.2.1 Well-Ordering Property of N: Every nonempty subset of N has a least (smallest) element.

1.2.2 Principle of Mathematical Induction :
Let S be a subset of N that possesses the two properties:
(1) The number 1 in S.
(2) For every k in N, if k in S, then k+ 1 in S.
Then we have S = N.

* Proof 1.2.1 implies 1.2.2

1.3.8 Theorem The set N X N is denumerable

1.3.10 Theorem The following statements are equivalent:
(a) S is a countable set.
(b) There exists a surjection of N onto S.
(c) There exists an injection of S into N.

1.3.11 Theorem The set Q of all rational numbers is denumerable.

1.3.12 Theorem If A_m is a countable set for each m in N, then the union A = $\bigcup_{m=1}^{\infty} A_m$ is countable.

* Proof of Theorem 1.3.12

1.3.13 Cantor's Theorem If A is any set, then there is no surjection of A onto the power set of A, the Set of all subsets of A. [not proof]

2.1.9 Theorem If a in R is such that $0 \leq a < e$ for every $e > 0$, then $a = 0$.

* Proof of Theorem 2.1.9

Arithmetic-Geometric Mean Inequality: If $a, b \geq 0$, then $\sqrt{ab} \leq \frac{a+b}{2}$; equality holds iff ab = 0.

*Proof of Arithmetic-Geometric Mean Inequality.

2.2.3 Triangle Inequality If a; b in R, then $|a + b| \leq |a| + |b|$; equality holds iff $ab \geq 0$.

*Proof of Triangle Inequality.

2.2.4 Corollary If a; b in R, then $| |a| - |b|| \leq | a – b|$; equality holds iff $ab \geq 0$.

*Proof of Triangle Inequality corollary 2.2.4.

2.3.3 Lemma A number u is the supremum of a nonempty subset S of R if and only if u satisfies the conditions:
(1) $s \leq u$ for all s in S,
(2) if $v < u$, then there exists s in S such that $v < s$
(2a) if $0 < e$, then there exists s in S such that $u - e < s$
(2b) if n in N, then there exists s in S such that $u - \frac{1}{n} < s$

*The proof of 2.3.3 and of analogous results for infimums and the discussion above this lemma are important.

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2.3.6 The Completeness Property of R Every nonempty set of real numbers that has an upper bound also has a supremum in R. (Statement, use but not proof)

Midterm Review : Analysis I - Spring 2019 – Dr. Smithies. Our midterm is Wed Oct 23rd.

2.4.1 Examples
Suppose that A and B are nonempty subsets of R that satisfy the property: for all a in A and b in B, a ≤ b,
Then Sup(A) ≤ Inf(B). (The result and its proof are important.)

2.4.3 Archimedean Property If x in R, then there exists n in N such that x ≤ n.

2.4.4 Corollary If S = $\{\frac{1}{n} \mid n \in N\}$ then Inf(S) = 0.

2.4.5 Corollary If t > 0, there exists n in N such that 0 < 1/n < t.

2.4.6 Corollary If y > 0, there exists n in N such that $n - 1 \leq y < n$.

*Proof of above results 2.4.3 and 2.4.5 the Archimedean Properties of R.

2.4.8 The Density Theorem If x and y are any real numbers with x < y, then there exists a rational number r in Q such that x < r < y.

2.5.2 Nested Intervals Property of R: Let $I_n = [a_n, b_n]$ be a sequence of nested, closed intervals of R, with I_{n+1} contained in I_n then $I = \bigcap_{n=1}^{\infty} I_n$ is non-empty.

We actually showed that I = $[\alpha, \beta]$ where $\alpha =$ Sup(a_n) and $\beta =$ Inf (b_n), but you do not need to know this proof.

2.5.4 Theorem The set R of real numbers is not countable

*Proof of 2.5.4

Corollary: the set R\Q of irrational numbers is uncountable.

*Proof of above corollary of 2.5.4

3.1.4 Uniqueness of Limits A sequence in R can have at most one limit.

*Proof of Theorem 3.1.4

3.1.9 Theorem Let X = (x_n) be a sequence of real numbers and let m in N. Then the m-tail $X_m = (x_{m+n})$ of X converges if and only if X converges. In this case, lim X_m = lim X.

*Proof of Theorem 3.1.9

3.2.2 Theorem A convergent sequence of real numbers is bounded.

*Proof of Theorem 3.2.2

3.2.3 Theorem (a) Let X = (x_n) and Y = (y_n) be sequences of real numbers that converge to x and y, respectively, and let c in R. Then the sequences X +Y, X – Y, XY, and cX converge to x + y, x – y, xy, and cx, respectively.
(b) If X = (x_n) converges to x and Z = (z_n) is a sequence of nonzero real numbers that converges to z and if z ≠0, then the quotient sequence X/Z converges to x/z.

The above results are called the Algebra of Limits. Their proofs are illustrative, but are not required.

3.2.4 Theorem If X = (x_n) is a convergent sequence of non-negative real numbers, then lim$(x_n) \geq 0$.

*Proof of Theorem 3.2.4

Midterm Review : Analysis I - Spring 2019 – Dr. Smithies. Our midterm is Wed Oct 23rd.

3.2.5 Theorem If X = (x_n) and Y = (y_n) are convergent sequences of real numbers and if $x_n \le y_n$ for all n in N, then $\lim(x_n) \le \lim(y_n)$.

*Proof of Theorem 3.2.5

3.2.7 Squeeze Theorem Suppose that X = (x_n), Y = (y_n), and Z = (z_n) are sequences of real numbers such that $x_n \le y_n \le z_n$ for all n in N, and that lim X = lim Z. Then Y is convergent and lim Y = lim X = lim Z.

The proof of the squeeze theorem is interesting but is not required.

3.2.9 Theorem Let the sequence X = (x_n) converge to x. Then the sequence $(|x_n|)$ of absolute values converges to $|x|$.

*Proof of Theorem 3.2.9

3.2.10 Theorem Let X = (x_n) be a sequence of non-negative real numbers that converges to x. Then the sequence $(\sqrt{x_n})$ of positive square roots converges and its limit is $\sqrt{x}$.

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*Proof of Theorem 3.2.10

3.2.11 Theorem Let X = (x_n) be a sequence of positive real numbers such that L = lim $(\frac{x_{n+1}}{x_n})$ exists. If L < 1, then X converges and lim(X) = 0 .

In the homework we showed if the ratio limit L > 1, then X diverges. The case L = 1 is inconclusive. Both the constant sequence (1) and the divergent sequence (n) have ratio limit L = 1. The ratio limit result is important but the proof is not.

3.3.2 Monotone Convergence Theorem A monotone sequence of real numbers is convergent if and only if it is bounded. Further, if X = (x_n) is increasing and bounded above, then lim X = sup(x_n) and if Y = (y_n) is decreasing and bounded below, then lim(Y) = inf(y_n).

*Theorem 3.3.2 (MCT) and its proof are very important.

3.4.2 Theorem If a sequence X = (x_n) of real numbers converges to a real number x, then any subsequence $X' = (x_{n_k})$ of X also converges to x.

*Proof of Theorem 3.4.2.

3.4.4 Theorem Let X = (x_n) be a sequence of real numbers, Then the following are equivalent:
(i) The sequence X does not converge to x in R.
(ii) There exists an $e_0 > 0$ such that for any k in N, there exists n_k in N such that $n_k \ge k$ and $|x_{n_k} - x| \ge e_0$.
(iii) There exists an $e_0 > 0$ and a subsequence $X' = (x_{n_k})$ of X such that for all k in N, $|x_{n_k} - x| \ge e_0$.

*Proof of Theorem 3.4.4. The next result is an immediate consequence of 3.4.4 and 3.2.2.

3.4.5 Divergence Criteria If a sequence X = (x_n) of real numbers has either of the following properties, then X is divergent.
(i) X has two convergent subsequences whose limits are not equal.
(ii) X is unbounded.

3.4.7 Monotone Subsequence Theorem If X = (x_n) is a sequence of real numbers, then there is a subsequence of X that is monotone.

Midterm Review : Analysis I - Spring 2019 – Dr. Smithies. Our midterm is Wed Oct 23rd.

3.4.8 The Bolzano-Weierstrass Theorem A bounded sequence of real numbers has a convergent subsequence.

*Theorem 3.4.8. is an immediate consequence of Theorem 3.4.7 and the MCT. You do not need to know the alternative proof using the Nested Interval Property.

3.4.11 Theorem If (x_n) is a bounded sequence of real numbers, then the following statements for a real number x^* are equivalent.
(a) $x^* = \text{limsup}(x_n)$.
(b) If e > 0, there are at most a finite number of n in N such that x^* + e < x_n, but an infinite number of n in N such that x^* - e < x_n.
(c) If u_m = sup(x_n : n ≥ m); then x^* = inf(u_m) = lim(u_m).
(d) If S is the set of subsequential limits of (x_n) then x^* = sup S.

* Also, be able to state the analogous characterizations of $x_* = \text{liminf}(x_n)$.

3.5.3 Lemma If X = (x_n) is a convergent sequence of real numbers, then X is a Cauchy sequence.

*Proof of Theorem 3.5.3

3.5.4 Lemma A Cauchy sequence of real numbers is bounded.

*Proof of Theorem 3.5.4

3.5.5 Cauchy Convergence Criterion A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

*Proof of 3.6.5

3.5.8 Theorem Every contractive sequence is a Cauchy sequence, and therefore is convergent.

You do not need to know the proof of Theorem 3.5.8 or the estimates on rates of convergence in Theorem 3.5.10.