**Calculus I - Test 2 Review - Fall 2017 - Dr. Smithies**

Test Two is **Friday October 13th**. Unless you have a University excused absence, you may incur a late penalty on a make up test. So please make every effort to be in the class on time and ready to work on this day.

If you have a documented reason why you would like to arrange your test with Student Access Service, you need to contact them. Their website is http://www.kent.edu/sas. Please let me know if you are testing through SAS since I need to email them the test.

Test Two will cover sections 2.6, 2.7, 3.1, 3.2, 3.3, 3.4 and 3.5 of our book. Sections 1.1, 1.2, and much of section 1.6 are prerequisites for this class, but they are needed in section 3.4 (graphing). The assigned homework for the material on test two was the odd problems in the problem sets below. Note that although this test is not explicitly over the test one material, you will need to know the test one material. In particular, you should be able to quickly and correctly apply the power, product, quotient and chain rules in calculating derivatives.

| **Section** | **Topic** | **Problems** | **Section** | **Topic** | **Problems** |
|---|---|---|---|---|---|
| 2.6 | Implicit Diff | 1-28 | 2.7 | Related Rates | 1-30 |
| 3.1 | Max, Mins | 1-24 | 3.2 | Mean Value Thm | 1-24 |
| 3.3 | Graph Shapes | 1-38 | 3.4 | Curve Sketching | 1-34 |
| 3.5 | Optimization | 1-31 | | | |

These homework problems and the worked examples from class and the book is the material to review for the test. **Definitions** are important potential test questions.

**Computing Derivatives**

You must be able to quickly and accurately calculate derivatives. You should know the derivatives of the most commonly used functions and all of our derivative rules. These are summarized in the next table.

| Function | Derivative | $y$ | $y'$ |
|---|---|---|---|
| $cf(x)$ ($c$ constant) | $cf'(x)$ | $c$ ($c$ constant) | $0$ |
| $f(x)+g(x)$<br>$f(x)-g(x)$ | $f'(x)+g'(x)$<br>$f'(x)-g'(x)$ | $\sin(x)$ | $\cos(x)$ |
| $y=u(x)v(x)$ | $y'=u'(x)v(x)+v'(x)u(x)$ | $\cos(x)$ | $-\sin(x)$ |
| $y=\frac{N(x)}{D(x)}$ | $y'=\frac{N'(x)D(x)-D'(x)N(x)}{[D(x)]^2}$ | $\cot(x)$<br>$\tan(x)$<br>$\csc(x)$<br>$\sec(x)$ | $-\csc^2(x)$<br>$\sec^2(x)$<br>$-\csc(x)\cot(x)$<br>$\sec(x)\tan(x)$ |
| $y=f\circ g(x)=f(g(x))$ | $y'=f'(g(x))g'(x)$ | $x^n$ | $nx^{n-1}$ |

You should also know the derivative of commonly used functions whose derivatives are given by the power rule. For example, for $c$ a constant

$$(\sqrt{x})'=\frac{1}{2\sqrt{x}},\quad (\frac{1}{x})'=\frac{-1}{x^2},\quad (c)'=0,\quad \text{and } (cx)'=c.$$

**Implicit Differentiation.**

We use *implicit differentiation* when the variables $x$ and $y$ are related by an equation like $x^3+y^3 = 6xy$ where it is not easy to express $y$ as an explicit function of $x$. With this method we can find the rate of change in $y$ with respect to $x$, $y' = \frac{dy}{dx}$, without solving for $y$ explicitly.

The idea is to recognize that $y = y(x)$, i.e., that $y$ is implicitly defined by $x$. We take the derivative of both sides of this equation using the chain rule:

$$\frac{d}{dx}[f(y)] = \frac{d}{dy}[f(y)]\frac{dy}{dx} = \frac{d}{dy}[f(y)]y'.$$

That is, the derivative of a function of $y$ is the usual derivative with respect to $y$ times the chain rule factor $y'$. Using this we can solve for $y'$ without ever solving for $y$ in terms of $x$.

**Related Rates problems**

These are story problems where you are given the rate of change with respect to time of some functions (Given $\frac{dx_1}{dt}, \cdots, \frac{dx_k}{dt}$) You are asked to find the how some other function $y$ changes with respect to time when the functions have some give value. That is find $\frac{dy}{dt}$ when $x_1, x_2, \cdots, x_k$ and $y$ have some particular value.

You solve these problems by using the geometry in the story problem to find an equation which relates the given and unknown variables $x_1, \cdots, x_k$ and $y$. Then use implicit differentiation with respect to $t$ to relate the given rates to the one you are solving for. Substitute in the known information to solve for the unknown derivative.

**Interpretations of the First Derivative**

The *difference quotient* of $f(x)$ at $x = a$ is $\frac{f(a+h)-f(a)}{h}$. It is the slope of the secant line between the points $(a, f(a))$ and $(a + h, f(a + h))$ on the graph of $f(x)$. The limit of the difference quotient as $h$ goes to 0 is the *derivative* of $f(x)$ at $a$, $f'(a)$. That is,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

The slope of the tangent line to the graph of $f(x)$ at $(a, f(a))$ is the derivative of $f(x)$ evaluated at $x = a$, $f'(a)$.

The physical interpretations of difference quotients and derivatives are as follows: The difference quotient of $f(x)$ at $a$ is the *average rate of change* of $f(x)$ over the interval $[a, a+h]$; The derivative to $f(x)$ at $a$ is the *instantaneous rate of change* of $f(x)$ at $a$.

By definition, a function $f(x)$ is *increasing* if $x_1 < x_2$ implies that $f(x_1) < f(x_2)$. That is, plugging in a greater input value yields a greater output. Similarly $f(x)$ is *decreasing* if $x_1 < x_2$ implies that $f(x_1) > f(x_2)$. That is, plugging in a greater input yields a smaller output. A function is increasing exactly when its derivative is positive and decreasing exactly when its derivative is negative. To see this, notice that for any $h \neq 0$, the difference quotient is positive if and only if the function is increasing. In summary,

$$f(x) \text{ is increasing if and only if } f'(x) > 0 \text{ and } f(x) \text{ is decreasing if and only if } f'(x) < 0$$

**Mean Value Theorem**

We covered Rolle's Theorem and the Mean Value Theorem. Our main use for this theorem will not be clear until later in the semester.

## Maxima and Minima

The *extrema* of a function are its local and global maxima and minima. Local maxima and minima have to occur where the derivative is either zero (horizontal tangent) or is undefined. The $x$ values $c$ such that $f'(c) = 0$ or $f'(c)$ Does Not Exist and any endpoints of the domain of $f(x)$ are called *critical values of* $f(x)$. We calculate the extrema of $f(x)$ on $[a, b]$ as follows.

(1) Calculate $f'(x)$.
(2) Solve for the critical values of $f(x)$ on $[a, b]$, i.e., $c$ such that $f'(c) = 0$ or DNE.
(3) List these critical values in increasing order $a = c_1 \le c_2 \le \cdots \le c_n = b$.
(4) Plug the critical values into the function to get $f(c_1), f(c_2), \cdots, f(c_n)$.
(5) Classify each $c_i$ by comparing $f(c_i)$ to $f(c_{i-1})$ and $f(c_{i+1})$.

## Interpretations of the Second Derivative.

Let $f(x)$ be a function with first and second derivatives $f'(x)$ and $f''(x)$. Since $f''(x)$ is the derivative of $f'(x)$, we know $f'(x)$ is increasing if and only if $f''(x)$ is positive. The first derivative $f'(x)$ is increasing exactly when the graph of $f(x)$ is *concave upward*. Combining these observations, we conclude that $f''(x)$ is positive exactly when the graph of $f(x)$ is concave upward. Similarly, $f''(x)$ is negative exactly when the graph of $f(x)$ is *concave downward*.

**Remark:** It is also useful to remember that the derivative does not always exist. The derivative is defined as the limit of the difference quotient and limits can fail to exists. In particular, the derivative does not exist at a place where the function is *not continuous*, or *has a corner* or *has a vertical tangent*. The derivative also does not exist at endpoints of the domain because the difference quotient does not have both both a right and left limit at these points.

## Graphing Techniques

We learned how to sketch the graph of a function $f(x)$ by analyzing the following features when they are reasonably easy to find.

(a) Domain and if easy to see, Range; (b) $x$ and $y$ intercepts; (c) symmetry (odd or even) and periodicity (trig functions); (d) asymptotes and holes; (e) monotonicity (increasing or decreasing) and maxima and minima; (f) concavity and inflection points; (g) Shape of graph.

We use the first and second derivatives to calculate when a graph is Increasing/Decreasing, Concave up/down, or has a Maxima/Minima/Inflection Point. Recall that $f' > 0$ is equivalent to $f$ is increasing, meaning when $a < b$, $f(a) < f(b)$. Similarly $f' < 0$ is equivalent to $f$ is decreasing, meaning when $a < b$, $f(b) < f(a)$. A function has a maximum where it switches from increasing to decreasing and a minimum where it switches from decreasing to increasing. A function $f$ is concave up exactly when $f'' > 0$. This is equivalent to saying $f'$ is increasing and so the graph of $f$ will resemble part of a parabola opening upward.

## Optimization

These are story problems where the goal is optimize (maximize or minimize) some quantity. Typically, the quantity being optimized depends on several variable. You must use the story problem and pre-calculus to identify relationships between your variables and then substitute out all but one of the variables. This reduces the problem to finding the $x$-values where a function $f(x)$ has its maximum or minimum value.