Calculus I -Spring 2018 - Dr. Smithies

Practice Test Four

(1) For each function $f(x)$ below:
(a) prove that $f(x)$ is one-to-one on its given domain,
(b) state the range of $f(x)$,
(c) calculate the inverse $f^{-1}(x)$,
(d) give the domain of $f^{-1}(x)$ and
(e) give the range of $f^{-1}(x)$ and

(1A) $f(x) = \frac{3x+2}{x-1}$ for $x \neq 1$
(1B) $f(x) = \frac{1}{x}$ for $x \neq 0$
(1C) $f(x) = 2x^2 + 8x + 1$ for $x \geq -2$

(2) In each of the following equations, find all solutions x.
(2A) $5^{3x+1} = 10$
(2B) $5^{2x+5} = 7^{x-1}$
(2C) $\log_5(x^2 + 3x) = 2$

(3) Calculate the derivatives of the following functions.
(3A) $f(x) = x^5 + 5^x - e^x + \ln(x) + 5$
(3B) $f(x) = e^{\frac{x^3+3x}{5x^4+3x}}$
(3C) $f(x) = \ln(\sin^2(x))$
(3D) $f(x) = \log_7(5x^6 + 6x^5)$
(3E) $f(x) = x \ln x$

(4) Calculate the following integrals.
(4A) $\int e^x - \frac{e^{\sqrt{x}}}{2\sqrt{x}}\, dx$
(4B) $\int \frac{x^7+x}{x^8+4x^2+1}\, dx$
(4C) $\int_1^2 \frac{1}{x}\, dx$
(4D) $\int 7^x + x^7 + 7\, dx$

(5) Calulate the requested information
(5A) $\lim_{x\to\infty} x - \ln(1+x)$
(5B) Sketch the region enclosed by the vertical lines $x = -1$ and $x = 1$ and the curves

$$f(x) = 2^x \text{ and } g(x) = (\frac{1}{2})^x$$

(5C) The absolute minimum of xe^{-x}

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Practice Test Four - Solution

(1) For each function $f(x)$ below: (a) prove that $f(x)$ is one-to-one on its given domain, (b) state the range of $f(x)$, (c) calculate the inverse $f^{-1}(x)$, (d) give the domain of $f^{-1}(x)$ and (e) give the range of $f^{-1}(x)$.

(1A) $f(x) = \frac{3x+2}{x-1}$ for $x \neq 1$

The function $f(x)$ is defined for every real number except $x = 1$. Assume a and b are real numbers which are not equal to each other and neither of which is 1. Assume further $f(a) = f(b)$. Then

$$\frac{3a+2}{a-1} = \frac{3b+2}{b-1}$$

This is

$$(3a+2)(b-1) = (3b+2)(a-1).$$

That is, $3ab - 3a + 2b - 2 = 3ab + 2a - 3b - 2$. This implies that $-3a + 2b = 2a - 3b$. We can write this as $3(b-a) = -2(b-a)$. If $a \neq b$ then $a - b \neq 0$ and this would imply that $3 = -2$. Clearly this isn't true and so we must have $a = b$. This exactly says that $a = b$. So, f is one-to-one on the given domain. The range of f is all real numbers except $y = 3$ because $y = 3$ is a horizontal asymptote which is approached but not crossed.

Let $y = \frac{3x+2}{x-1}$. Then $y(x-1) = 3x+2$ and so $yx - y = 3x+2$. It follows that $yx - 3x = 2+y$ and so $x(y-3) = 2+y$. Dividing we get, $x = f^{-1}(y) = \frac{2+y}{y-3}$. Thus, $f^{-1}(x) = \frac{x+2}{x-3}$. The domain of $f^{-1}(x)$ is every number except $x = 3$. The range includes all real numbers except $y = 1$ because $y = 1$ is a horizontal asymptote which is approached but not crossed.

(1B) $f(x) = \frac{1}{x}$ for $x \neq 0$

You know the graph of $y = \frac{1}{x}$ and it satisfies the horizontal line test. So f is one-to-one on its domain. We also know from the graph that the range is all real numbers except $y = 0$.

You can calculate the inverse of f by solving $y = \frac{1}{x}$ for x. To do this, multiply both sides by x and divide both sides by y. Then $x = f^{-1}(y) = \frac{1}{y}$. In other words, $f^{-1}(x) = \frac{1}{x}$. This makes sense because applying $f(x)$ and then $f^{-1}(x)$ to a number x takes the reciprocal of x twice which is just x. The domain and range of $f^{-1}(x)$ are both all non-zero real numbers.

(1C) $f(x) = 2x^2 + 8x + 1$ for $x \geq -2$

Here $f(x) = 2x^2 + 8x + 1$ and so $f'(x) = 4x + 8$. Thus $f'(x) \geq 0$ for $x \geq -2$. Since $f(x)$ is monotone increasing, it is one-to-one. Moreover, $f(-2) = 8 - 16 + 1 = -7$. Since f is increasing on its domain, the range of $f(x)$ is $y \geq -7$.

To get the inverse, we have to solve $y = 2x^2 + 8x + 1$ for x. We can do this by applying the quadratic formula to $2x^2 + 8x + 1 - y = 0$. This gives us

$$x = \frac{-8 \pm \sqrt{64 - 4(2)(1-y)}}{2(2)} = \frac{-8 \pm \sqrt{64 - 8(1-y)}}{4} = -2 + \sqrt{\frac{7+y}{2}}.$$

Notice that we chose the positive branch of the square root because of domain of $f(x)$ is $x \geq -2$. That is, $f^{-1}(x) = -2 + \sqrt{\frac{7+x}{2}}$. The domain of the inverse is $\frac{7+x}{2} \geq 0$ which is $x \geq -7$. Since the square root takes on all non-negative values, the range of the inverse is $y \geq -2$.

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(2) In each of the following equations, find all solutions x.

(2A) $5^{3x+1} = 10$

We apply the natural log to both sides. Then $\ln(5^{3x+1}) = \ln(10)$. By the log rules, this is $(3x+1)\ln(5) = \ln(10)$. So, $3x+1 = \frac{\ln(10)}{\ln(5)}$. Thus,

$$x = \frac{\frac{\ln(10)}{\ln(5)} - 1}{3} = .1435588$$

(2B) $5^{2x+5} = 7^{x-1}$

Again, since we are solving for exponents, we apply the natural log to both sides. This gives us, $(2x+5)(\ln(5)) = (x-1)\ln(7)$. That is, $\ln(5)(2x)+\ln(5)(5) = \ln(7)(x)+(-1)\ln(7)$. So, $\ln(5)(2x) - \ln(7)(x) = -\ln(7) - \ln(5)(5)$. This implies that

$$x = \frac{-[\ln(7) + 5\ln(5)]}{2\ln(5) - \ln(7)} = \frac{-9.993099}{1.272965} = -7.85025.$$

(2C) $\log_5(x^2+3x) = 2$

We start by using both sides as exponents on 5. This gives us $5^{\log_5(x^2+3x)} = 5^2$. That is, $x^2+3x = 25$ or $x^2+3x-25 = 0$. We use the quadratic formula to solve for x.

$$x = \frac{-3 \pm \sqrt{109}}{2} = 3.72015 \text{ or } -6.72015.$$

Both solutions work.

(3) Calculate the derivatives of the following functions.

(3A) $f(x) = x^5 + 5^x - e^x + \ln(x) + 5$

Differentiating term-by-term we get $f'(x) = 5x^4 + \ln(5)5^x - e^x + \frac{1}{x}$.

(3B) $f(x) = e^{\frac{x^3+3x}{5x^4+3x}}$

This is a chain rule problem. The outside function is $h(x) = e^x$ and its derivative is $h'(x) = e^x$. The inside function is $g(x) = \frac{x^3+3x}{5x^4+3x}$. By the quotient rule,

$$g'(x) = \frac{(3x^2+3)(5x^4+3x) - (20x^3+3)(x^3+3x)}{(5x^4+3x)^2}.$$

That is,

$$g'(x) = \frac{[15x^6+15x^4+9x^3+9x] - [20x^6+60x^4+3x^3+9x]}{(5x^4+3x)^2} = \frac{-5x^6-45x^4+6x^3}{(5x^4+3x)^2}.$$

The chain rule gives us

$$f'(x) = h'(g(x))g'(x) = (e^{\frac{x^3+3x}{5x^4+3x}})(\frac{-5x^6-45x^4+6x^3}{(5x^4+3x)^2}).$$

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(3) Calculate the derivatives of the following functions.

(3C) $f(x) = \ln(\sin^2(x))$

This is a chain rule problem. The outside function is $h(x) = \ln(x)$. Its derivative is $h'(x) = \frac{1}{x}$. The inside function is $g(x) = \sin^2(x)$. Its derivative $g'(x) = 2\sin(x)\cos(x)$ is given by the chain rule. So,

$$f'(x) = h'(g(x))g'(x) = \frac{1}{\sin^2(x)}(2\sin(x)\cos(x)) = \frac{2\cos(x)}{\sin(x)} = 2\cot(x).$$

(3D) $f(x) = \log_7(5x^6 + 6x^5)$

This is a chain rule problem. The outside function is $h(x) = \log_7(x)$. Its derivative is $h'(x) = \frac{1}{\ln(7)x}$. The inside function is $g(x) = 5x^6+6x^5$. Its derivative is $g'(x) = 30x^5+30x^4 = 30x^4(x+1)$. By the chain rule,

$$f'(x) = h'(g(x))g'(x) = \frac{1}{\ln(7)(5x^6 + 6x^5)}(30x^4(x+1)) = \frac{30x^4(x+1)}{\ln(7)(5x^6 + 6x^5)}.$$

(3E) $f(x) = x\ln x$

This is a product rule problem. $f'(x) = (1)(\ln(x)) + (x)(\frac{1}{x}) = \ln(x) + 1$.

(4) Calculate the following integrals.

(4A) $\int e^x - \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx$

First note that $\int e^x - \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx = \int e^x\,dx - \int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx$. The first term integrates directly. The second term requires substitution. Let $u = \sqrt{x}$. The $du = \frac{1}{2\sqrt{x}}\,dx$. So,

$$\int e^x\,dx - \int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx = \int e^x\,dx - \int e^u\,du = e^x - e^u + c = e^x - e^{\sqrt{x}} + c.$$

(4B) $\int \frac{x^7+x}{x^8+4x^2+1}\,dx$

Let $u = x^8 + 4x^2 + 1$. Solving the derivative of u, we get $du = 8x^7 + 8x\,dx$. So, $\frac{1}{8}du = x^7 + x\,dx$. Thus,

$$\int \frac{x^7 + x}{x^8 + 4x^2 + 1}\,dx = \frac{1}{8}\int \frac{du}{u} = \frac{1}{8}\ln(x^8 + 4x^2 + 1) + c.$$

(4C) $\int_1^2 \frac{1}{x}\,dx$

$$\int_1^2 \frac{1}{x}\,dx = \ln(x)|_{x=1}^{x=2} = \ln(2) - \ln(1) = .693147181 - 0 = .693147181.$$

(4D) $\int 7^x + x^7 + 7\,dx$

$$\int 7^x + x^7 + 7\,dx = \frac{7^x}{\ln(7)} + \frac{x^8}{8} + 7x + c.$$

(5A) $\lim_{x\to\infty} x - \ln(1+x)$

Consider $f(x) = e^{x-\ln(1+x)} = \frac{e^x}{e^{\ln(1+x)}} = \frac{e^x}{1+x}$. By, L'Hospital's rule for the form $\frac{\infty}{\infty}$, $\lim_{x\to\infty} \frac{e^x}{1+x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty$.

(5B) Sketch the region enclosed by the curves

$$f(x) = 2^x \text{ and } g(x) = (\frac{1}{2})^x$$

and the vertical lines $x = -1$ and $x = 1$.

The region looks like a bowtie. It has vertices $(-1, f(-1))$, $(-1, g(-1))$, $(0, 1)$, $(1, f(1))$, $(1, g(1))$. That is $(-1, \frac{1}{2})$, $(-1, 2)$, $(0, 1)$, $(1, 2)$, $(1, \frac{1}{2})$.

(5C) The absolute minimum of xe^{-x}

Let $f(x) = xe^{-x}$. By the product rule $f'(x) = (1)(e^x) - (x)(e^x) = e^x(x-1)$. We see that the derivative of $f(x)$ is negative for $x < 1$ and positive for $x > 1$. So, the only critical point $x = 1$ is an absolute minimum.