Calculus I - Test 4 Review - Spring 2018 - Dr. Smithies

Test Four is **Friday April 27th**. The test will focus on inverses, natural and general exponentials and logarithm functions, and the graphs and limits of these functions. These are discussed in sections 5.1 through 5.5, and 5.8 of our book. The assigned homework for these sections was the problems with solutions in the back of your book in the problem sets:

(5.1) 1-31, (5.2) 1-60, (5.3) 1-43 and (5.4) 1-38, (5.5) 1-17, (5.8) 1-39.

These problems and the worked examples from class and the book could be on the test. We will cover 5.6 Inverse Trig functions after test 4. Section 5.6 is hyperbolic trig functions. We do not cover these.

Main Topics

(5.1) Inverses Let $f(x)$ be a function from a set A to a set B. If it exists, the *inverse of* f is a function f^{-1} from B to A such that

$$f^{-1}(f(x)) = x \text{ for all } x \text{ in } A \quad \text{and} \quad f(f^{-1}(y)) = y \text{ for all } y \text{ in } B.$$

A function $f(x)$ which maps the set A onto the set B only has an inverse if f is *one-to-one*. This means if $f(a) = f(b)$ then $a = b$. Equivalently, if $a \neq b$ then $f(a) \neq f(b)$. In other words, no two different input values have the same output under $f(x)$. This is necessary in order for $f^{-1}(x)$ to be a well-defined function.

Some options for showing that $f : A \to B$ is one-to-one are:

(1) Simplify the equation $f(a) = f(b)$ to show that $a = b$.

(2) Sketch the graph of $f(x)$, show f satisfies the horizontal line test.

(3) Calculate the derivative of $f(x)$ and show that f is either monotone increasing or monotone decreasing.

(4) Calculate the inverse of $f(x)$ by solving $y = f(x)$ for $x = f^{-1}(y)$ and check that $f^{-1}(y)$ is a well-defined function.

To compute the inverse of $f(x)$, solve $y = f(x)$ for x. Then $x = f^{-1}(y)$. You can substitute an x for every y in this function $f^{-1}(y)$ to get $f^{-1}(x)$. The point of doing this is just that we are used to x being the independent variable.

The graphs of $f(x)$ and $f^{-1}(x)$ are reflections of each other about the line $y = x$.

You can calculate the derivative of f^{-1} from the derivative of $f(x)$. The formula is

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

This comes from taking the derivative of both sides of the equation $f(f^{-1}(x)) = x$. However, this formula is not used very often. Instead you would ususually calculate the derivative of f^{-1} directly.

(5.2) Natural Log The exponential function which is most often used in scientific applications is the natural base

$$e = 2.718281828\cdots.$$

Because it comes up so often we write $\log_e(x) = \ln(x)$. That is, when the base is e, we write the logarithm as ln. It is called the *natural logarithm*.

Since it is a logarithm, it satisfies all of the log rule. In particular, for any non-negative numbers x and y, and any real number r

$$\ln(xy) = \ln(x) + \ln(y), \quad \ln(\frac{x}{y}) = \ln(x) - \ln(y), \quad \ln(x^r) = r\ln(x).$$

(5.2) Natural Log Instead of defining $\ln(x)$ as the inverse of e^x, you can define the natural logarithm of x as the area below the function $\frac{1}{t}$ and above the t-axis and between the vertical line $t = 1$ and $t = x$ That is, $\ln(x) = \int_1^x \frac{1}{t}\,dt$. The point of doing this is that it follows immediately from the Fundamental Theorem of Calculus that

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

The proof of this derivative formula is not important to us, but the derivative rule is. Notice that this derivative formula immediately implies the anti-derivative rule

$$\int \frac{1}{x}\,dx = \ln|x| + c.$$

Any logarithm can be calculated in terms of the natural log. Specifically, the *base change formula* says

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}.$$

That is, for any base b, the function $\log_b(x)$ is just the natural log function times the constant $\frac{1}{\ln(b)}$. This base change formula comes from using the logarithm properties. Namely, the defining relationship

$$y = b^x \text{ is equivalent to } \log_b(y) = x$$

says that in order to calculate $\log_b(y)$ you need to solve the equation $y = b^x$ for x. To do this, take the natural log of both sides of $y = b^x$. Using the log rules, this give us

$$\ln(y) = \ln(b^x) \text{ which is } \ln(y) = x\ln(b) \text{ so } x = \frac{\ln(y)}{\ln(b)}.$$

It follows from the base change formula that

$$\frac{d}{dx}[\log_b(x)] = \frac{d}{dx}[\frac{\ln(x)}{\ln(b)}] = \frac{1}{\ln(b)}[\frac{d}{dx}(\ln(x))] = \frac{1}{\ln(b)}(\frac{1}{x}) = \frac{1}{\ln(b)x}$$

(5.3) Natural Exponential The function e^x is just the exponential function for the special base e. It satisfies all the exponential rules.

$$e^x e^y = e^{x+y} \text{ and } \frac{e^x}{e^y} = e^{x-y} \text{ and } (e^x)^r = e^{rx}.$$

The inverse derivative formula, or equivalently, the chain rule applied to $\ln(e^x) = x$ implies that

$$\frac{d}{dx}[e^x] = e^x$$

In other words, e^x is its own derivative. That is one of the reasons it is so important. It follows from this, and the base change formula that

$$\frac{d}{dx}[b^x] = \ln(b)b^x.$$

Of course, all of these derivative formulas define integral formluas. Namely,

$$\int e^x\,dx = e^x + c \text{ and } \int b^x\,dx = \frac{b^x}{\ln(b)} + c.$$

(5.4) General Logs and Exponentials

Let b be a fixed positive constant. The function b^x is defined first on the positive integers. That is, for each $n = 1, 2, 3, \cdots$

$$b^n = bb \cdots b \text{ a total of } n \text{ factors of } b.$$

From this defition, it is easy to check that

$b^x b^y = b^{x+y}$

$\frac{b^x}{b^y} = b^{x-y}$

$(b^x)^r = b^{rx}$

Using these rules it is easy to see that for any positive integer n

$b^0 = 1$, $b^{-n} = \frac{1}{b^n}$ and $b^{\frac{1}{n}} = \sqrt[n]{b}$, the n-th root of b.

These observations extend the definition of b^x for any positive constant b to allow any rational number $x = \frac{n}{m}$. Namely, $b^{\frac{n}{m}} = \sqrt[m]{b^n} = (\sqrt[m]{b})^n$. This extends to any real number x as an exponent because every real number is a limit of a sequence of rational numbers.

Let b be a fixed positive number. By definition, $\log_b$ is the inverse of the exponential function b^x. It is defined by the relationship

$$y = b^x \text{ is equivalent to } \log_b(y) = x.$$

This means that when y is input to the function $\log_b$, the output $\log_b(y)$ is the exponent which b must be raised to, in order to get y.

Notice that the function b^x has domain of all real numbers and range of all non-negative numbers. Since $\log_b$ is the inverse of b^x, its range is all real numbers and **the domain of** $\log_b(x)$ **is the non-negative numbers.** In other words, you cannot take the log of 0 or a negative number because there is no exponent x for which b^x is not a positive number.

The above rules for exponential imply analogous rules for logarithms. Namely, for any non-negative numbers x and y, and any real number r

$\log_b(xy) = \log_b(x) + \log_b(y)$

$\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$

$\log_b(x^r) = r \log_b(x)$

Some important special cases to keep in mind are $\log_b(b) = 1$ and $\log_b(1) = 0$.

CAUTION There is no formula to simplify the log of a sum or difference. That is, neither $\log_b(x + y)$ nor $\log_b(x - y)$ simplify in term of $\log_b(x)$ or $\log_b(y)$.

What the log does is converts product and quotients to sums and differences (laws 1 and 2 above) and it converts exponents to multiples.

(5.5) Exponential growth and decay

These are story problems involving exponentials. We saw that the information that the rate of growth is proportional to the amount of material present translates into the equation $y' = ky(t)$, where $y(t)$ is the amount present at time t. This equation translates into $y(t) = Ce^{kt}$. We learned that $C = y(0)$ is the initial amount and k is either given or solved for using logs. We looked at radiative decay, population growth, temperature change and continuously compounded interest. Remember that the return R when principle P is invested for t years at a continuously compounded interest rate of r is $R = Pe^{rt}$.

(5.8) L'Hospital's Rule

L'hospital's rule is a rule for calculating limits. It says if $\frac{f(a)}{g(a)}$ has the form $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$ then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

We can also use algebra to rewrite other special forms until L'Hospitals rule does apply. If $\lim_{x\to a} h(x)$ has the form $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞ or ∞^0 then $h(x)$ can be algebraically manipulated until it is a limit problem which has the form $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$.

It is helpful to remember the following limits.

If $0 < a < 1$	$\lim_{x\to -\infty} a^x = \infty$	$\lim_{x\to +\infty} a^x = 0$
If $a > 1$	$\lim_{x\to -\infty} a^x = 0$	$\lim_{x\to +\infty} a^x = \infty$
In particular,	$\lim_{x\to -\infty} e^x = 0$	$\lim_{x\to +\infty} e^x = \infty$
Finally,	$\lim_{x\to 0^+} \ln(x) = -\infty$	$\lim_{x\to +\infty} \ln(x) = \infty$

We learned the following derivative and integral formulas in Chapter 5.

Derivatives: (Where c is a constant.)

$(a^x)' = \ln(a)a^x$
$(\log_a(x))' = \frac{1}{\ln(a)x}$
$(e^x)' = he^x$
$(\ln(x)' = \frac{1}{x}$
$(e^{h(x)})' = h'(x)e^{h(x)}$
$(\ln(h(x))' = \frac{h'(x)}{h(x)}$

Anti-derivatives: (Where c is a constant.)

$\int a^x \, dx = \frac{a^x}{\ln(a)} + c$
$\int \frac{1}{\ln(a)x} \, dx = \log_a(x) + c$
$\int e^x \, dx = e^x + c$
$\int \frac{1}{x} \, dx = \ln(|x|) + c$
$\int \frac{h'(x)}{h(x)} \, dx = \ln(h(x)) + c$