## Excursions in Modern Math - Fall 2019 - Dr. Smithies

### Counting Methods

(1) Sum whole numbers from 1 to $N$. (See page 20).

The total when you add the whole numbers from 1 to $N$ is

$$1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}.$$

For example, $1 + 2 + 3 + 4 + 5 = 15$, but instead of adding you could get that from using $N = 5$ in this formula. This gives you the total $\frac{5(6)}{2} = \frac{30}{2} = 15$.

(2) Subtasks - Be able to take a given task and describe it as several non-overlapping smaller subtasks. For example, the task of creating a license plate number like ABC 123 could be described as 6 subtasks. The first 3 are pick a capital letter, the last 3 are pick a digit.

(3) Addition Rule - Sometimes a task breaks up as subtask-1 through subtask-N and you complete the task by doing **exactly one** of the subtasks. In other words, you do the task by doing subtask-1 **OR** subtask-2 **OR** $\cdots$ **OR** subtask-N. The total number of ways to do a task is the **sum** of the number of ways to do all the independent subtasks. For example, if your task is to pick a character which is either an upper or lower case letter or an odd digit, then there are $26 + 26 + 5 = 57$ ways to complete this task.

(4) Multiplication Rule - Sometimes a task breaks up as subtask-1 through subtask-N and you complete the task by doing **all** of the subtasks. In other words, you do the task by doing subtask-1 **AND** subtask-2 **AND** $\cdots$ **AND** subtask-N. The total number of ways to do a task is the **product** of the number of ways to do all the independent subtasks. For example, if your task is to pick a four digit pin number then there are $(10)(10)(10)(10) = 10,000$ ways to complete this task.

(5) Factorial - By definition, $0! = 1$ and $1! = 1$. In general,

$$N! = N\,(N-1)! = N\,(N-1)\cdots(2)\,(1).$$

That is, $N!$ means count backward from $N$ to 1 and multiply all of these factors together.

For example, the number of ways to arrange 5 objects is $5! = (5)(4)(3)(2)(1) = 120$ ways. This comes from breaking the task of finding all arrangements into the subtasks (a) Choose the first object [5 ways to do], (b) Choose the second object [4 ways to do],..., (e) Choose the 5th place object [1 way to do]. This type of Multiplication Rule problem comes up often enough that it is convenient to just know that the number of ways to arrange $N$ objects is $N!$.

**Counting Methods**

(6) Permutations - The number of *rearrangements* of a collection of $k$ objects, chosen from a set of $n$ objects is ${}_nP_k$. By definition,

$$ {}_nP_k = \frac{n!}{(n-k)!} $$

For example, the number of ways to pick a president and a vice president from the 7 candidates $\{A, B, C, D, E, F, G\}$ is

$$ {}_7P_2 = \frac{7!}{(7-2)!} = \frac{7!}{5!} = \frac{(7)(6)(5)(4)(3)(2)(1)}{(5)(4)(3)(2)(1)} = (7)(6) = 42. $$

This really comes from the multiplication rule. The task of picking the president and vice president breaks down to two subtasks. First pick the president (7 ways) then pick someone else to be the vice president (6 ways).

(7) Combinations - The number of *unordered* set of $k$ objects, chosen from a set of $n$ objects is ${}_nC_k$. By definition,

$$ {}_nC_k = \frac{n!}{(n-k)!k!} = \frac{{}_nP_k}{k!}. $$

This comes from taking the number of ordered $k$-out-of-$n$-objects collections ${}_nP_k$ and dividing by the number of ways to arrange $k$ objects, $k!$.

For example, the number of ways to select 2 committee members from the 7 candidates $\{A, B, C, D, E, F, G\}$ is

$$ {}_7C_2 = \frac{7!}{(7-2)!2!} = \frac{7!}{5!2!} = \frac{(7)(6)(5)(4)(3)(2)(1)}{[(5)(4)(3)(2)(1)][(2)(1)]} = \frac{(7)(6)}{(2)(1)} = \frac{42}{2} = 21. $$

Another way to look at this is you could select a president and vice president from the 7 candidates $\{A, B, C, D, E, F, G\}$. We saw in the last example there are 42 ways to do this. But then if the two people selected are just committee members, then the choices $(A, B)$ is the same as the choice $(B, A)$. The 42 ordered choices have to be divided by the number of ways to arrange 2 objects.

(8) Pascal's Triangle is easy to quickly write out. The outside entries are 1s and the interior entries are the sum of the entries above it. It turns out that the entries of this triangle are the combinations $C_k^n$. The entries in the $n$-th row are $n$ choose $0, 1, \cdots, n$.

$$
\begin{array}{lccccccccc}
\text{Row } 0 & & & & & 1 & & & & \\
\text{Row } 1 & & & & 1 & & 1 & & & \\
\text{Row } 2 & & & 1 & & 2 & & 1 & & \\
\text{Row } 3 & & 1 & & 3 & & 3 & & 1 & \\
\text{Row } 4 & 1 & & 4 & & 6 & & 4 & & 1
\end{array}
\qquad
\begin{array}{lccccccccc}
\text{Row } 0 & & & & & C_0^0 & & & & \\
\text{Row } 1 & & & & C_0^1 & & C_1^1 & & & \\
\text{Row } 2 & & & C_0^2 & & C_1^2 & & C_2^2 & & \\
\text{Row } 3 & & C_0^3 & & C_1^3 & & C_2^3 & & C_3^3 & \\
\text{Row } 4 & C_0^4 & & C_1^4 & & C_2^4 & & C_3^4 & & C_4^4
\end{array}
$$