

Excursions - Math 11008 -002 - Fall 2019 - Dr. Smithies
Practice Voting Theory Exam
Section A - Voting Methods

	Number of Votes	10	6	5
Consider the Preference Schedule:	1st Choice	<i>A</i>	<i>B</i>	<i>C</i>
	2nd Choice	<i>C</i>	<i>D</i>	<i>B</i>
	3rd Choice	<i>D</i>	<i>C</i>	<i>D</i>
	4th Choice	<i>B</i>	<i>A</i>	<i>A</i>

- (1) How many voters participated? $10 + 6 + 5 = 21$
- (2) How many candidates are in the election? **4**
- (3) Who (if anyone) is the majority candidate? **There is none.**
- (4) Who (if anyone) is the Condorcet candidate? **C**
 C beats A (11 to 10), C beats B (15 to 6) and C beats D (15 to 6). So, C beats all opponents head-to-head.
- (5) What does this example prove about the relationship between a majority candidate and a Condorcet candidate?
A Condorcet candidate need not be a majority candidate.
 The example shows that an election with a Condorcet candidate need not have a majority candidate. Recall that a majority candidate wins all head-to-head comparisons and so the majority candidate, if it exists, is always the Condorcet candidate. The reverse of this is not true.
- (6) Who wins in a head-to-head comparison of A and D?
D wins 11 to 10 over A
- (7) How many points are available when this election is decided by pairwise comparisons?
 $\frac{4(3)}{2} = 6$
- (8) Find the Borda total for A.
 $A = (4)(10) + 2(0) + 1(11) = 51.$
- (9) Who is first eliminated in plurality-with-elimination?
D has no first place votes. **C** is eliminated next with 4 first place votes.
- (10) Who wins a head-to-head comparison of A and B?
B wins 11 to 10 over A.
- (11) Is the winner of the election by plurality-with-elimination the same as the winner of the a head-to-head comparison of A and B?
Yes We know D and the C are eliminated. Plurity applied to the remaining 2 candidates gives the same winner as head-to-head comparison (or any other voting method, since there are only two candidates).

- (12) Suppose D is found to be ineligible to run for election. Give the preference schedule after D drops out of the election.

	Number of Votes	10	6	5
	1st Choice	<i>A</i>	<i>B</i>	<i>C</i>
	2nd Choice	<i>C</i>	<i>C</i>	<i>B</i>
	3rd Choice	<i>B</i>	<i>A</i>	<i>A</i>

**Excursions - Math 11008 -002 - Fall 2019 - Dr. Smithies
Practice Voting Theory Exam**

- (13) How many pairwise comparisons must be made in this election (after D was removed)?
 $\frac{3(2)}{2} = 3.$
- (14) Who (if anyone) is the Condorcet candidate after D is removed?
C. Since head-to-head comparisons are determined by which candidate is higher on the preference schedule, the results of the head-to-head comparisons do not change when other candidates drop out. Thus, C beats A (11 to 10) and C beats B (15 to 6).
- (15) Who wins by the pairwise comparison voting method?
C The Condorcet candidate always win the pairwise comparison method.
- (16) What is the total number of points available when the Borda voting method is used on the above 3 candidate election?
126 points. There a total of 21 voters. The first place votes are worth 3 points, second place is worth 2 points and third place is worth 1 point. The total available is $3(21) + 2(21) + 1(21) = 6(21) = 126$

Section B - Fairness Criteria

- (1) Which of the following fairness criteria are satisfied by the voting method of plurality?
(a) The majority criterion.
(b) The Condorcet criterion.
(c) The monotonicity criterion.
(d) The independence of irrelevant alternatives.
Plurality satisfies (a) and (c). The majority criterion is satisfied since if a candidate has a majority (i.e., more than half) of the available first place votes then this candidate has the plurality of the votes (i.e., more than any other candidate). Monotonicity is satisfied since if a candidate, X , has more first place votes than any opponent and in a re-vote the only candidate with any favorable changes is X , then X is still the plurality winner, because X still has more first place votes than any opponent. We've seen that Plurality violates the Condorcet criterion. The example with the marching bands has a candidate (Hula Bowl) who beats all other candidates head-to-head but the plurality winner is (Rose Bowl) not the Condorcet candidate. Another example showed that Plurality (and our other 3 voting methods) violate the independence of irrelevant alternatives.
- (2) Does every election have a majority candidate? (Y/N)
No. A majority candidate is a candidate with more than half the available first place votes. There isn't always such a candidate in every election.
- (3) Suppose an election is held and C wins. Then a re-vote is held and all changes are favorable to only C. If C loses which of our 4 voting method was used and why?
Plurality-with-elimination This is the only voting method we studied which violates the Monotonicity Fairness criterion.

**Excursions - Math 11008 -002 - Fall 2019 - Dr. Smithies
Practice Voting Theory Exam**

- (4) Which voting method satisfies all 4 Fairness criterion?

None. Arrow's Impossibility Theorem says there is no voting method which satisfies all 4 fairness criteria.

- (5) Which Fairness criterion is violated by all 4 of the voting methods we studied?

Independence of Irrelevant Alternatives There are voting methods which satisfy this but none of the 4 voting methods we studied do.

- (6) If a voting method satisfies the Majority criterion, does the candidate with the most first place votes always win?

No. The candidate with more than half the first place votes must win but the candidate with more first place votes than any opponent need not win unless they have more than half of the first place votes.

Section C - Ranking, Ties and Voting Theory

- (1) In an election with candidates A, B, C, D, and E and 77 voters, how many different preference ballots can be cast?

77 There are $5! = 120$ ways to rearrange the candidates. But there are only 77 voters. If all voters vote differently, there would be 77 different ballots.

- (2) Can an election with an odd number of voters end in a tie in all four of our voting methods? (Y/N)

Yes For example,

Number of Votes	3	3	3
1st Choice	A	B	C
2nd Choice	B	C	A
3rd Choice	C	A	B

Each candidate has exactly 1 win and 1 loss in head-to-head competition. So pairwise comparison is a tie. Each candidate has a Borda total of $3(3) + 2(3) + 1(3) = 18$. So the Borda method ends in a tie. Each candidate has 3 first place votes. So both plurality end in a tie. Plurality-with-elimination also end in a tie. There is no candidate who we can eliminate with a fewest number of first place votes and there is no candidate with a majority of the first place votes who can be declared a winner.

- (3) Can a head-to-head comparison be a tie when the number of voters is odd?

No. In the comparison X vs Y, suppose X and Y have the same number of votes, M . Then the total number of votes cast is $M + M = 2M$.

Excursions - Math 11008 -002 - Fall 2019 - Dr. Smithies
Practice Voting Theory Exam

- (4) Can an election which has a majority candidate result in a tie by the Borda voting method?

Yes. Consider the preference schedule

Number of Votes	3	2	2
1st Choice	A	C	C
2nd Choice	B	A	B
3rd Choice	C	B	A

The Borda totals are: $A = 3(3) + 2(2) + 1(2) = 15$ $B = 3(0) + 2(5) + 1(2) = 12$ and $C = 3(4) + 2(0) + 1(3) = 15$. So, A and C tie for first place under the Borda Method. Note the head-to-head tie breaker makes C the winner 4 to 3.

- (5) Can an election with a majority candidate end in a tie by the plurality-with-elimination voting method?

No. This method picks the majority candidate as the winner, if a majority candidate exists.

- (6) Can an election with a majority candidate end in a tie by the pairwise comparison voting method?

No. Since a majority candidate is also a Condorcet candidate, the pairwise comparison voting method picks the majority candidate as the winner, if a majority candidate exists.

- (7) In an election with N voters and 7 candidates, what is the total number of Borda points available?

$21N$ There are

$$7N + 6N + \cdots + 2N + N = (7 + 6 + 5 + 4 + 3 + 2 + 1)N = 21N$$

total points in the election.

- (8) In an election with N voters and 7 candidates, what is the highest Borda point total a candidate can get?

$7N$ The highest total would happen if one candidate received all N 1st place votes.

- (9) In an election with N voters and 7 candidates, what is the highest pairwise comparison point total a candidate can get?

6. The highest total would happen if a candidate beats all 6 of his opponents head-to-head.

- (10) Give the extended pairwise ranking for the following preference schedule.

Number of Votes	3	2	2	1
1st Choice	A	C	C	D
2nd Choice	B	A	B	C
3rd Choice	D	D	A	B
4th Choice	C	B	D	A

C,A,B,D The comparisons are A beats B (5 to 3), C beats A (5 to 3), A beats D (7 to 1), C beats B (5 to 3), B beats D (5 to 3) and C ties D (4 to 4). The points are $A = 2$, $B = 1$, $C = 2.5$ and $D = .5$. The Extended pairwise ranking is C,A,B,D.

Excursions - Math 11008 -002 - Fall 2019 - Dr. Smithies
Practice Voting Theory Exam

- (11) Give the Plurality-with-Elimination ranking for the following preference schedule.

Number of Votes	3	2	2	2
1st Choice	<i>A</i>	<i>C</i>	<i>C</i>	<i>D</i>
2nd Choice	<i>B</i>	<i>A</i>	<i>B</i>	<i>A</i>
3rd Choice	<i>D</i>	<i>D</i>	<i>A</i>	<i>B</i>
4th Choice	<i>C</i>	<i>B</i>	<i>D</i>	<i>C</i>

A,C,D,B. *B* is eliminated first with no 1st place votes; then *D* is eliminated with only 2 1st place votes. The two votes of *D* transfer to *A*, who then defeats *C* 5 to 4.

- (12) Give the Plurality ranking for the above preference schedule.

C,A,B,D With 4,3,2 and 0 1st place votes, respectively, the plurality ranking is C,A,D,B.