

Weighted Voting Review

Overview: Chapter 2 is a study of weighted voting systems. Players can have more than one vote. They vote on Yes or No motions, not on a list of options. You should know the basic definitions about weighted voting systems, how to find winning coalitions, identify critical players and give the Banzhaf power index for each player.

A generic weighting voting system looks like $[q : w_1, w_2, \dots, w_n]$. There are n players in this system. We call them P_1, \dots, P_n . For example, Player one, P_1 , has w_1 votes. The first number q is the *quota*. It is the minimum number of votes a motion must have to pass. The *total number of votes* is $V = w_1 + \dots + w_n$. Recall that to be valid, the quota must satisfy

$$\frac{V}{2} < q \leq V.$$

That is, the quota must be more than half the total votes but not more than all the total votes.

A *coalition* is a collection of players. The *weight of a coalition* is the total votes of the players in the coalition. A coalition is called a *winning coalition* if its weight equals or exceeds the quota. A player in a winning coalition is *critical* if the coalition becomes losing when that player is dropped from it. The *critical uses* of player P_k is B_k the total number of winning coalitions in which player P_k is critical. The *total critical usages* is $T = B_1 + \dots + B_n$ is the total of each players critical uses.

Player P_k is a *dictator* if its total votes meets or exceeds the quota. Generally we list the number of votes in non-increasing order ($w_1 \geq w_2 \geq \dots \geq w_n$). In this case, the only possible dictator is Player one, P_1 . Player one is a dictator exactly when $w_1 \geq q$. Player P_k has *veto power* if P_k is not a dictator and the remaining player do not have enough votes to meet the quota without the votes of Player k . That is,

$$w_k < q \text{ and } V - w_k < q,$$

where V is the total votes. Player P_k is a *dummy* if it has no critical uses. That is, $B_k = 0$.

The *Banzhaf Power Distribution* of the weighted voting system $[q : w_1, w_2, \dots, w_n]$ is calculated as follows.

- (1) List all winning coalitions.
- (2) Find the critical uses B_1, B_2, \dots, B_n of each player.
- (3) Let $T = B_1 + B_2 + \dots + B_n$ be the total critical usages.
- (4) The *power index* of player P_k is $\beta_k = \frac{B_k}{T}$

So your power index is your critical uses divided by the total critical usages.

Except for rounding, you should have the sum of the power indexes is 100%. That is, $\beta_1 + \beta_2 + \dots + \beta_n = 100$. The critical uses in step (2) above can be found two different ways. You could go through each winning coalition one at a time and underline a player if they are critical (dropping just them makes the coalition losing). Then you get B_k by counting how many time player P_k was underlined. Alternatively, you could use the formula

$$B_k = [\text{number of winning coalitions with } P_k] - [\text{number of winning coalitions without } P_k]$$