# A note on polar decomposition based Geršgorin-type sets

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Dedicated to Richard S. Varga, on his 80-th birthday.

**Abstract:** Let $B \in \mathbb{C}^{n\times n}$ denote a finite-dimensional square complex matrix. In [6] and [7], Professor Varga and I introduced Geršgorin-type sets which were developed from Singular Value Decompositions (SVDs) of $B$. In this note, our work is extended by introducing the polar SV-Geršgorin set, $\Gamma^{\mathrm{PSV}}(B)$. The set $\Gamma^{\mathrm{PSV}}(B)$ is a union of $n$ closed discs in $\mathbb{C}$, whose centers and radii are defined in terms of the entries of a polar decomposition $B = Q|B|$. The set of eigenvalues of $B$, $\sigma(B)$, is contained in $\Gamma^{\mathrm{PSV}}(B)$.

**keywords:** polar decomposition, Geršgorin type sets, normal matrices, eigenvalue estimates.

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## 1 Introduction and Background

In the paper [6], we developed a theoretical analysis of how, for $n \leq \infty$, the set of all singular value decompositions of an $n \times n$ complex matrix $B$ can be used to construct a Geršgorin-type set, $\Gamma^{\mathrm{SV}}(B)$, which contains the eigenvalues of $B$. In [7], we narrowed our focus to the case of $n$ finite, and we showed how the entries of any fixed SVD $V\Sigma W^*$ of a non-zero $B \in \mathbb{C}^{n\times n}$ could be used to form a Geršgorin-type set, $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$. We showed that the set $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$ contains the eigenvalues of $B$ and that if $B$ is normal, the SVD of $B$ can be constructed in such a way that $\Gamma^{\mathrm{NSV}}(V\Sigma W^*)$ equals $\sigma(B)$, the set of eigenvalues of $B$.

Let $B \in \mathbb{C}^{n\times n}$ be non-zero. Every SVD of $B$, $V\Sigma W^*$ also defines a polar decomposition of $B = Q|B|$ (both described below) and vice versa. This raises the natural question of how a polar decomposition can be used to construct a Geršgorin-type set. In this note, we define the Geršgorin-type set, $\Gamma^{\mathrm{PSV}}(B)$ which is constructed from the entries of the unique polar decomposition which satisfies $\mathrm{Ker}(Q) = \mathrm{Ker}(|B|)$. The set $\Gamma^{\mathrm{PSV}}(B)$ is a union of $n$ closed discs in the complex plane, and it contains the spectrum, $\sigma(B)$, of $B$. Moreover, we show that if $B$ is normal and non-singular (i.e., $B$ is "maximally compatible with our methods"), then the polar factor $Q$ is diagonal and invertible. In this case, the $k$-th SV-polar Geršgorin set $\Gamma_k^{\rm PSV}(B)$ is just the $k$-th Geršgorin set of $|B|$, $\Gamma_k(|B|)$, but with the the center $|B|_{kk}$ rotated by the diagonal element $Q_{kk}$.

## 2 Notation

Let $w = (w_j)_{j=1}^n$ denote a column vector in $\mathbb{C}^n$, where $n$ is finite. Denote the supremum and Euclidean norms on $w \in \mathbb{C}^n$ by

$$\|w\|_\infty := \max_{j=1,\cdots,n}\{|w_j|\}, \text{ and } \|w\|_2 := \sqrt{<w,w,>} := \sqrt{\Sigma_{j=1}^n |w_j|^2},$$

respectively. The relationship between these norms is

$$\|w\|_\infty \le \|w\|_2, \tag{1}$$

since if the maximal modulus of the components of $w$ occurs in, say, the $k$-th component, then $\|w\|_\infty = \sqrt{|w_k|^2} \le \sqrt{\Sigma_{j=1}^n |w_j|^2} = \|w\|_2$.

Let $B \in \mathbb{C}^{n\times n}$ denote a non-zero, finite-dimensional square complex matrix, and let $B^* = \overline{B}^T$ be the adjoint of $B$. A *singular value decomposition of* $B$ is an expression of $B$ as a product $B = V\Sigma W^*$, where $V$ and $W$ are square unitary matrices, and $\Sigma := \mathrm{Diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$, where $\sigma_1 \ge \sigma_2 \ge, \cdots, \ge \sigma_n \ge 0$. Note that if $B = V\Sigma W^*$, then the square of the absolute value of $B$ is $|B|^2 := B^*B = W\Sigma V^* V \Sigma W^* = W\Sigma^2 W^*$. Since $W$ is assumed to be unitary (i.e., $W^*W = WW^* = I \in \mathbb{C}^{n\times n}$), the columns of $W$ must form an orthonormal basis of $\mathbb{C}^n$, consisting of eigenvectors of $|B|^2$. Similarly, the columns of $V$ form an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of $|B^*|^2 = BB^*$. The matrix $B$ is normal (i.e., $B^*B = BB^*$) if and only if there exists a SVD of $B$ such that, for each $j = 1, \cdots, \mathrm{Rank}(B)$, the $j$-th column of $V$ is a complex rotation times the $j$-th column of $W$.

A *polar decomposition* of a matrix $B \in \mathbb{C}^{n\times n}$ is an expression of $B$ as a product $B = Q|B|$, where the factor $|B| = \sqrt{B^*B}$ is the (unique) absolute value of $B$, and the factor $Q$ is a partial isometry from the range of $|B|$ onto the range of $B$. More precisely,

$$Q^*Qx = x, \quad \forall x \in \mathrm{Ran}(|B|) \qquad \text{and} \qquad QQ^*y = y, \quad \forall y \in \mathrm{Ran}(B).$$

The additional constraint

$$\mathrm{Ker}(Q) = \mathrm{Ker}(|B|) \tag{2}$$

makes $Q$ (and hence the polar decomposition of $B$) unique. When we say *the* polar decomposition of $B$, we will always mean the unique factorization $B = Q|B|$ which satisfies Equation (2).

Let $V\Sigma W^*$ denote any fixed SVD of a non-zero matrix $B \in \mathbb{C}^{n\times n}$. Then $|B| = W\Sigma W^*$ and $Q = VW^*$ defines a polar decomposition of $B$. The components of an SVD of $B$ can be used to express $B$ as a sum of rank one operators in

$\mathbb{C}^{n\times n}$, and this allows us to construct the unique polar factorization $B = Q|B|$ which satisfies Equation (2). More precisely, let

$$B = V\Sigma W^* = \begin{bmatrix} | & \cdots & | \\ \phi_1 & \cdots & \phi_n \\ | & \cdots & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_n \end{bmatrix} \begin{bmatrix} - & \bar{\psi}_1^T & - \\ \vdots & \vdots & \vdots \\ - & \bar{\psi}_n^T & - \end{bmatrix} \tag{3}$$

denote a fixed SVD of $B$, and set $r := \mathrm{Rank}(B)$. For any fixed $j$, $\sigma_j \phi_j \psi_j^*$ is the rank one $n \times n$ matrix $\sigma_j\, \phi_j (\bar{\psi}_j)^T$. It takes any $x \in \mathbb{C}^n$ to the complex multiple $\sigma_j < x, \psi_j >$ of the vector $\phi_j$. Thus,

$$B = \Sigma_{l=1}^n \sigma_l \phi_l \psi_l^* = \Sigma_{l=1}^r \sigma_l \phi_l \psi_l^*.$$

The factors, $Q = \Sigma_{l=1}^r \phi_j \psi_j^*$ and $|B| = \Sigma_{l=1}^r \sigma_j \psi_j \psi_j^*$ give the desired unique polar factorization of $B$.

## 3 Our Results

Now we define a Geršgorin-type set which is constructed from the entries of the polar decomposition $B = Q|B|$. In order to make our set look as similar to a standard Geršgorin set as possible, we assume that all of the diagonal entries of the polar factor $Q$ are non-zero. This assumption can be viewed as a convenience, in the sense that the eigenvalue estimates given by our polar Geršgorin set are only intended to be useful when $Q$ is essentially diagonal.

**Definition 1** *Let $B$ be an non-zero $n \times n$ complex matrix, and let $B = Q|B|$ denote the polar decomposition of $B$. Assume that the diagonal entries of $Q$ are non-zero. Set $D := \mathrm{Diag}(Q)$, and $E := Q - D$. Define the SV-polar parameter to be*

$$\epsilon_{PSV} = \sqrt{n}\|B\|_2\|E\|_2.$$

*For each $k = 1, \cdots, n$, denote the $k$-th deleted row sum of the $|B|$ by $R_k(|B|) := \Sigma_{j\neq k}|B|_{kj}$, and define the $k$-th SV-polar bound to be*

$$R_k^{\mathrm{PSV}} = \frac{\epsilon_{PSV} + R_k(|B|)}{|Q_{kk}|}.$$

*Let the $k$-th SV-polar Geršgorin set be*

$$\Gamma_k^{\mathrm{PSV}}(B) := \{z \in \mathbb{C} \ : \ |z - |B|_{kk}/\overline{Q}_{kk}| \leq R_k^{\mathrm{PSV}}\},$$

*and define the SV-polar Geršgorin set as*

$$\Gamma^{\mathrm{PSV}}(B) := \cup_{k=1}^n \Gamma_k^{\mathrm{PSV}}(B).$$

**Theorem 2** *Let $B$ be an $n \times n$ complex matrix, and let $B = Q|B|$ denote the polar decomposition of $B$. Assume $Q_{kk} \neq 0$, for all $k = 1, \cdots, n$. Define $\Gamma^{\mathrm{PSV}}(B)$ as in Definition 1 and let $\sigma(B)$ denote the spectrum of $B$. Then,*

$$\sigma(B) \subset \Gamma^{\mathrm{PSV}}(B).$$

Proof: Let $B = Q|B|$, with $Q$ satisfying the given assumptions. Set $Q := D + E$, as in Definition 1. Let $Bx = \lambda x$, with $\|x\|_2 = 1$. Then, $|B|x = \lambda Q^* x$. Set $y = \lambda E^* x$. Define $\epsilon$ by $\epsilon := \frac{\|y\|_\infty}{\|x\|_\infty}$. Equation (1) implies that

$$\|y\|_\infty = \|\lambda E^* x\|_\infty = |\lambda| \|E^* x\|_\infty \leq \|B\|_2 \|E^* x\|_\infty \leq \|B\|_2 \|E^* x\|_2 \leq \|B\|_2 \|E\|_2.$$

Since $\|x\|_2 = 1$, $\|x\|_\infty \geq \frac{1}{\sqrt{n}}$. Thus, $\frac{1}{\|x\|_\infty} \leq \sqrt{n}$. It follows that

$$\epsilon := \frac{\|y\|_\infty}{\|x\|_\infty} \leq \sqrt{n} \|B\|_2 \|E^*\|_2 =: \epsilon_{PSV}. \tag{4}$$

Equating $k$-components in the vector equality $|B|x = \lambda Q^* x$ and using the relation $Q^* = D^* + E^*$ gives us:

$$\Sigma_{j=1}^{n} |B|_{kj} x_j = \lambda \overline{Q}_{kk} x_k + y_k.$$

Thus, for all $k = 1, \cdots, n$,

$$(|B|_{kk} - \lambda \overline{Q}_{kk}) x_k = \Sigma_{j \neq k} |B|_{kj} x_j - y_k.$$

We apply the above equation for the fixed index $k$ such that $|x_k| = \|x\|_\infty$. As in the usual Geršgorin proof (see [8]) the triangle inequality gives us

$$||B|_{kk} - \lambda \overline{Q}_{kk}| \leq \frac{|y_k|}{|x_k|} + \Sigma_{j \neq k} ||B|_{kj}| \frac{|x_j|}{|x_k|} \leq \epsilon + \Sigma_{j \neq k} ||B|_{kj}| = \epsilon + R_k(|B|).$$

Since

$$||B|_{kk} - \lambda \overline{Q}_{kk}| = (|\overline{Q}_{kk}|)(||B|_{kk} / \overline{Q}_{kk} - \lambda|) = (|Q_{kk}|)(||B|_{kk} / \overline{Q}_{kk} - \lambda|),$$

and $\epsilon \leq \epsilon_{PSV}$ (see Equation (4)), we have shown

$$||B|_{kk} / \overline{Q}_{kk} - \lambda| \leq \frac{\epsilon_{PSV} + R_k(|B|)}{|Q_{kk}|} =: R_k^{\mathrm{PSV}}. \tag{5}$$

Equation (5) says that $\lambda \in \Gamma_k^{\mathrm{PSV}}(B)$. $\square$

In the extreme case, $Q$ is diagonal and non-singular. Since $Q$ is also unitary $1/\overline{Q}_{kk} = Q_{kk}$ and $|Q_{kk}| = 1$. Moreover, in this case $E := Q - D = 0$ and so $\epsilon_{PSV} = 0$. Thus, $R_k^{\mathrm{PSV}} := \frac{\epsilon_{PSV} + R_k(|B|)}{|Q_{kk}|} = R_k(|B|)$, the $k$-th deleted row sum of $|B|$. These observations establish to the following theorem.

**Theorem 3** *Let $B$ be a non-singular $n \times n$ complex matrix, and let $B = Q|B|$ denote the polar decomposition of $B$. Assume $Q$ is diagonal. Then, the SV-polar set reduces to:*

$$\Gamma^{\mathrm{PSV}}(B) = \cup_{k=1}^{n} \{z \in \mathbb{C} \ : \ |z - |B|_{kk} Q_{kk}| \leq R_k(|B|)\}.$$

Stated differently, if we assume that $B$ is non-singular and normal, then there exists a SVD of $B$, $V\Sigma W^*$, which satisfies $Q = VW^*$ is diagonal and invertible. For the polar decomposition of $B$ constructed from this SVD, Theorem 3 says that the $k$-th SV-polar Geršgorin set $\Gamma_k^{\mathrm{PSV}}(B)$ is just the $k$-th Geršgorin set of $|B|$, $\Gamma_k(|B|)$, but with the the center $|B|_{kk}$ rotated by $Q_{kk}$.

It should be pointed out that this note is only intended to complete our theoretical approach to using factorizations of a matrix to develop Geršgorin-type estimates of its eigenvalues. That is, our previous work naturally raises the question of how a polar decomposition can be used to estimate eigenvalues, and this note presents an answer to this. However, the computational methods for constructing a polar decomposition require the construction of a SVD and our work in [7] gives a much more efficient and practical eigenvalue estimate, based on a SVD.

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