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Finite rank harmonic operator-valued functions

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Abstract

The purpose of this paper is to characterize when a harmonic function with values in the finite rank operators on a Hilbert space is expressible as a harmonic matrix-valued function. We show that harmonic function with values in the rank 1 normal operators is expressible as a harmonic matrix-valued function. We also prove that for any natural number, *n*, a harmonic function with values in the rank *n* non-negative operators is expressible as a matrix-valued function and we give examples showing that these decomposition theorems fail when various hypotheses are relaxed. © 2001 Elsevier Science Inc. All rights reserved.

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Throughout this paper, let $\mathscr{H}$ be a separable Hilbert space. Denote by $\mathrm{Kom}(\mathscr{H})$ the algebra of compact operators on $\mathscr{H}$ and let $\mathrm{Rank}^n(\mathscr{H}) \subseteq \mathrm{Kom}(\mathscr{H})$ denote the rank n operators on $\mathscr{H}$, where an operator A is *rank* n if the range of A and the orthogonal complement of the kernal of A are both of dimension less than or equal to n.

Recall that a function v from an open subset $\mathscr{U}$ of the complex plane to the bounded linear operators on $\mathscr{H}$, $\mathscr{L}(\mathscr{H})$, is called *harmonic* if the composition of v with every bounded linear functional on $\mathscr{L}(\mathscr{H})$ is a harmonic complex-valued function. For a function taking values in the compact operators, there are several simpler characterizations of when it is harmonic (cf. [1]). In particular, $v : \mathscr{U} \to \mathrm{Kom}(\mathscr{H})$ is harmonic if and only if for each orthonormal basis $\{\phi_i\}_{i=1}^{\infty}$ of $\mathscr{H}$, the functions $\langle v(z)\phi_i, \phi_i\rangle$ are harmonic.

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We say a harmonic function $v : \mathscr{U} \to \mathrm{Rank}^n(\mathscr{H})$ is *expressible as a harmonic $k \times k$ matrix-valued function* if there exists a set of orthonormal vectors $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$ in $\mathscr{H}$ such that for every $z \in \mathscr{U}$ the range and orthogonal complement of the kernel of the operator $v(z)$ are spanned by these k vectors. In other words, if we extend $\{\gamma_i\}_{i=1}^k$ to form an orthonormal basis $\{\gamma_i\}_{i=1}^\infty$ of $\mathscr{H}$, and express each $v(z)$ in this basis, then each $v(z)$ will have non-zero entries only in its upper $k \times k$ block and, as functions of z, the matrix entries of v will all be harmonic.

Those finite rank operator-valued functions which are expressible as matrix-valued functions form an important subclass in applications which involve perturbing an operator by a "smooth" finite rank operator-valued function. (See [2,4,5] for examples of finite rank perturbation problems.) The purpose of this note is to characterize when a finite rank operator-valued functions is expressible as a matrix-valued function. We begin by showing that a harmonic rank n operator-valued function with values in the non-negative operators is expressible as a harmonic $k \times k$ matrix-valued function for some $k \leqslant n$. The same proof establishes this result for harmonic rank n operator-valued function with values in the non-positive operators.

Lemma 1. *Let $\mathscr{U}$ be an open, connected, simply connected neighborhood in $\mathbb{C}$ and let $v : \mathscr{U} \to \mathrm{Rank}^n_{\mathscr{P}}(\mathscr{H})$ be a harmonic function taking values in the non-negative rank n operators. Then there exists a $k \leqslant n$ such that v can be expressed as a harmonic $k \times k$ matrix-valued functions. Moreover, the rank of each $v(z)$ is independent of z.*

Proof. The result is trivial if v is identically zero and so we assume it is not. Fix $z_0 \in \mathscr{U}$ such that $\mathrm{Rank}(v(z_0))$ is minimal and let k denote $\mathrm{Rank}(v(z_0))$. Let $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$ be an orthonormal basis for range of $v(z_0)$. Let $\mathscr{S} = \mathrm{Ker}(v(z_0)) = \mathrm{Range}(v(z_0))^{\perp}$. We will show for each $z \in \mathscr{U}$ and every $\beta \in \mathscr{S}$, $\langle v(z)\beta, \beta\rangle = 0$. It follows that v is expressible as a harmonic $k \times k$ matrix-valued function in the vectors $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$.

Let $\beta \in \mathscr{S}$. Then $\langle v(z)\beta, \beta\rangle$ is a non-negative harmonic function on $\mathscr{U}$ which vanishes at z_0. The mean value property for harmonic functions implies that there exists a positive real number ρ (depending on β) such that for any fixed r in the interval $[0, \rho]$,

$$\langle v(z_0)\beta, \beta\rangle = \frac{1}{2\pi}\int_0^{2\pi} \langle v(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta})\beta, \beta\rangle\,\mathrm{d}\theta.$$

Since $\langle v(z_0)\beta, \beta\rangle = 0$ and the integrand is a non-negative continuous function of θ, this shows that $\langle v(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta})\beta, \beta\rangle = 0$ for every $r \in [0, \rho]$ and every $\theta \in [0, 2\pi]$. In other words, $\langle v(z)\beta, \beta\rangle$ is identically zero on the disk $\Delta(z_0, \rho)$. Therefore $\langle v(z)\beta, \beta\rangle$ is identically zero on $\mathscr{U}$, since $\mathscr{U}$ is simply connected and $\langle v(z)\beta, \beta\rangle$ is a real-valued harmonic function.

Finally, because z_0 was chosen so that the rank of $v(z_0)$ is minimal as z varies over $\mathscr{U}$, we see from the above proof that the rank of $v(z)$ is necessarily k for each $z \in \mathscr{U}$. $\square$

One consequence of the previous lemma is that a harmonic function $v : \mathscr{U} \to \mathrm{Rank}^n_{\mathscr{P}}(\mathscr{H})$ from an open, connected, simply connected subset of the complex plane to the non-negative rank n operators necessarily has a harmonic conjugate on $\mathscr{U}$ with values in the rank n operators. Of course, we need the assumption that $\mathscr{U}$ is simply connected to establish the existence of a harmonic conjugate to v, however we do not actually need the hypothesis that $\mathscr{U}$ is simply connected in Lemma 1. We show this in the next theorem.

Theorem 1. *Let $\mathscr{U}$ be an open, connected neighborhood in $\mathbb{C}$ and let $v : \mathscr{U} \to \mathrm{Rank}^n_{\mathscr{P}}(\mathscr{H})$ be a harmonic function taking values in the non-negative rank n operators. Then there exists a $k \leqslant n$ such that v can be expressed as a harmonic $k \times k$ matrix-valued functions. Moreover, the rank of each $v(z)$ is independent of z.*

Proof. Let v be as above and fix a point a in $\mathscr{U}$ and an open disc $\Delta(a, r)$ in $\mathscr{U}$. We know from Lemma 1 that there is an orthonormal set of vectors $\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ such that the restriction of v to this disc, $v(z)|_{\Delta(a,r)}$ is expressible as a $k \times k$ matrix-valued harmonic function. We will show that for any $w \in \mathscr{U}$, there is an open neighborhood, W, of w in $\mathscr{U}$ on which $v(z)|_W$ can be expressed as a $k \times k$ matrix-valued harmonic function in these same vectors $\{\gamma_i\}_{i=1}^k$.

In view of Lemma 1, the existence of the neighborhood W of w will be established if we can show that there is an open connected, simply connected subset of $\mathscr{U}$ containing both a and w. Although the proof is elementary, for completeness we show this now. Since $\mathscr{U}$ is an open connected subset of $\mathbb{C}$, it is path-wise connected. Choose a finite length piece-wise smooth arc in $\mathscr{U}$, $A(a, w)$, which connects a and w. Then $A(a, w)$ is compact and the collection of open discs

$$\{\Delta(b, r_b) \subseteq \mathscr{U} \mid b \in A(a, w)\}$$

is an open cover of $A(a, w)$. Thus, we can choose a finite chain $\{\Delta(b_i, r_i)\}_{i=1}^n$ covering $A(a, w)$ such that $b_1 = a$, $b_n = w$, and $\Delta(b_i, r_i) \cap \Delta(b_{i+1}, r_{i+1}) \neq \emptyset$. The set $W = \bigcup_{i=1}^n \Delta(b_i, r_i)$ is an open, connected, simply connected subset of $\mathscr{U}$ containing both a and w. $\square$

If we apply Theorem 1, in the case of a non-negative rank one operator-valued harmonic function, $v : \mathscr{U} \to \mathrm{Rank}^1_{\mathscr{P}}(\mathscr{H})$, we get that $||v(z)||$ is a harmonic function on $\mathscr{U}$ and there exists a unit vector $\phi \in \mathscr{H}$ such that for each $z \in \mathscr{U}$

$$v(z) = ||v(z)||\langle\cdot\,, \phi\rangle\phi.$$

The following example shows that this conclusion fails if we relax the “smoothness” hypothesis from harmonic to infinitely differentiable.

Example 1. Let $\mathscr{U}$ be any open, connected set in $\mathbb{C}$. In this example we construct a function $v : \mathscr{U} \to \mathrm{Rank}^1_{\mathscr{P}}(\mathscr{H})$ which is C^∞ in x and y (where $z = x + \mathrm{i}y \in \mathscr{U}$) but which is not expressible as a 1×1 matrix-valued function. Choose disjoint open subsets W_1 and W_2 of $\mathscr{U}$ and non-empty compact subsets $K_1 \subset W_1$ and $K_2 \subset W_2$. By the C^∞ analog of Urysohn's lemma (cf. [3]) there are non-negative C^∞ functions f_1 and f_2 satisfying $f_i = 1$ on K_i and $f_i = 0$ outside W_i, for $i = 1, 2$. Define $v : \mathscr{U} \to \mathrm{Rank}^1_{\mathscr{P}}(\mathscr{H})$ by

$$v(z) = f_1(z)\langle \cdot\,, \phi_1\rangle\phi_1 + f_2(z)\langle \cdot\,, \phi_2\rangle\phi_2,$$

where ϕ_1 and ϕ_2 are orthogonal, non-zero vectors in $\mathscr{H}$.

The previous example shows that we cannot relax the notion of smoothness in Theorem 1. However, we can extend the class of operators for harmonic rank 1 operator-valued functions. We show this in the next theorem.

Theorem 2. *Let $\mathscr{U}$ be an open, connected neighborhood in $\mathbb{C}$ and let $v : \mathscr{U} \to \mathrm{Rank}^1_{\mathscr{N}}(\mathscr{H})$ be a harmonic function taking values in the normal rank 1 operators. Then there exists a unit vector $\phi \in \mathscr{H}$ such that if $u(z) = \langle v(z)\phi, \phi\rangle$, then*

$$v(z) = u(z)\langle \cdot\,, \phi\rangle\phi \quad \forall z \in \mathscr{U}.$$

Proof. The result is evident if $v \equiv 0$ and so we assume this is not the case. Moreover, the proof which extends Lemma 1 to Theorem 1 applies without change. Thus, we assume without loss of generality that $\mathscr{U}$ is simply connected.

First, assume $v : \mathscr{U} \to \mathrm{Rank}^1_{\mathscr{S}\mathscr{A}}(\mathscr{H})$ is a harmonic function taking values in the self-adjoint operators. Since v takes values in the rank 1 self-adjoint operators, at each $z \in \mathscr{U}$, $v(z)$ has a decomposition of the form

$$v(z) = \mu(z)\langle \cdot\,, \phi_z\rangle\phi_z$$

for some unit vector ϕ_z and some real number $\mu(z)$ (both depending on z). Let $\mathscr{U}^+ = \{z \in \mathscr{U} : \mu(z) > 0\}$ and $\mathscr{U}^- = \{z \in \mathscr{U} : \mu(z) < 0\}$. One of these sets has non-empty interior. Hence, we can fix a neighborhood $\mathscr{W}$ in $\mathscr{U}$ such that the restriction of v to $\mathscr{W}$ is either a function with values in the non-positive rank 1 operators or a function with values in the non-negative rank 1 operators. In either case, Theorem 1 implies that there exists a unit vector ϕ such that

$$v(z) = \lambda(z)\langle \cdot\,, \phi\rangle\phi \quad \forall z \in \Delta(z_0, r),$$

where $\lambda(z) = \langle v(z)\phi, \phi\rangle$. But then $u(z) = \lambda(z)\langle \cdot\,, \phi\rangle\phi$ defines a harmonic function on all of $\mathscr{U}$ which agrees with v on the neighborhood $\Delta(z_0, r)$. Since $\mathscr{U}$ is simply connected, $v = u$ on all of $\mathscr{U}$. This establishes the theorem in the case where v takes values in the self-adjoint rank 1 operators.

Now, assume that v takes values in the normal rank 1 operators. The real and imaginary parts of v are harmonic self-adjoint operator-valued functions. Moreover, since v takes values in the normal operators, its real and imaginary parts inherit the property of taking values in the rank 1 operators. Hence, we know that there exist unit vectors ϕ_1 and ϕ_2 and harmonic real-valued functions μ_1 and μ_2 such that $\mathrm{Re}(v) = \mu_1\langle\cdot\,,\phi_1\rangle\phi_1$ and $\mathrm{Im}(v) = \mu_2\langle\cdot\,,\phi_2\rangle\phi_2$. If ϕ_1 and ϕ_2 were not linearly dependent, then v would take values in the rank 2 operators, contrary to our assumption. Thus, there exists a harmonic complex-valued function λ on $\mathscr{U}$ such that

$$v(z) = \lambda(z)\langle\cdot\,,\phi\rangle\phi$$

on all of $\mathscr{U}$. $\square$

In the following example, we show that if the hypothesis that v takes values in the normal operators is dropped, then v may fail to be expressible as a harmonic matrix-valued functions.

Example 2. Let $\mathscr{U}$ be the open unit disc $\Delta(0,1)$ in $\mathbb{C}$. For each $n \in \mathbb{N}$, define $\lambda_n : \mathscr{U} \to \mathbb{R}$ to be the harmonic function $\lambda_n(z) = \mathrm{Re}(z^n)$. Fix an orthonormal basis $\{\phi_i\}_{i=1}^{\infty}$ of $\mathscr{H}$ and define $w : \mathscr{U} \to \mathscr{H}$ by

$$w(z) = \sum_{n=1}^{\infty} \lambda_n(z)\phi_n.$$

It is easy to check that w is a harmonic function from $\mathscr{U}$ to $\mathscr{H}$. Now fix any unit vector ϕ in $\mathscr{H}$ and define $v : \mathscr{U} \to \mathrm{Rank}^1(\mathscr{H})$ by

$$v(z) = \langle\cdot\,, w(z)\rangle\phi.$$

The function v is harmonic, since for any choice $\{\psi_j\}_{j=1}^{\infty}$ of orthonormal basis for $\mathscr{H}$, the diagonal matrix coefficients $\langle v(z)\psi_i, \psi_i\rangle = \langle\psi_i, w(z)\rangle\langle\phi, \psi_i\rangle$ are harmonic. Moreover, v takes values in the rank 1 operators and for each $z \in \mathscr{U}$, the range of $v(z)$ is spanned by ϕ. However, v is not expressible as a matrix-valued function. Indeed, there cannot exist vectors $\gamma_1, \ldots, \gamma_k$ which span $\mathrm{Ker}(v(z))^{\perp}$ for all $z \in \mathscr{U}$ since if $z_1 = \frac{1}{2}$, $z_2 = \frac{1}{3}, \ldots, z_{k+1} = \frac{1}{k+2}$, then $\mathrm{Ker}(v(z_i))^{\perp}$ is spanned by $w(z_i)$. The vectors $w(z_1), \ldots, w(z_{k+1})$ are linearly independent and hence cannot be spanned by the set $\{\gamma_1, \ldots, \gamma_k\}$. Thus, the analog of Theorem 2 fails when we drop the hypothesis that v takes values in the normal operators.

The previous example has implications for harmonic functions with values in the rank r self-adjoint operators for $r \geqslant 2$.

Example 3. Let $\mathscr{U}$, $\{\phi_n\}_{n=1}^{\infty}$, $\{\lambda_n(z)\}_{n=1}^{\infty}$ and $w(z)$ be as in the previous example. Let $\phi = \phi_1$ and let $v(z) = \langle\cdot\,, w(z)\rangle\phi$. Since v is harmonic, its adjoint function $v^*(z) = \langle\cdot\,,\phi\rangle w(z)$ is also harmonic. Hence,

$$u(z) = v(z) + v^*(z) = \langle\cdot\,, w(z)\rangle\phi + \langle\cdot\,,\phi\rangle w(z)$$

is a harmonic function with values in the rank 2, self-adjoint operators. However, there is no finite choice of vectors $\{\gamma_1, \ldots, \gamma_k\}$ in $\mathscr{H}$ such that for every $z \in \mathscr{U}$, the function $u(z)$ can be expressed in terms of these k vectors. Indeed, as z varies over the open unit disc in $\mathbb{C}$, it is easy to see that the ranges of the $u(z)$ cannot all be spanned by any fixed finite collection of vectors.

Since the function u in the previous example can also be regarded as a rank r, self-adjoint operator-valued function for any r greater than or equal to 2, this shows that Theorem 1 does not generalize to finite rank self-adjoint harmonic functions.

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