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Harnack's theorem for harmonic compact operator-valued functions

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Abstract

In this paper we show that harmonic compact operator-valued functions are characterized by having harmonic diagonal matrix coefficients in any choice of basis. We also give an example which shows that an operator-valued function with values outside the compact operators can have harmonic diagonal matrix coefficients in any choice of basis without being a harmonic operator-valued function. We use our harmonic matrix coefficients characterization to establish a Harnack's theorem for an increasing sequence of harmonic compact self-adjoint operator-valued functions and we show that this Harnack's theorem need not hold when the compactness restriction is dropped. © 2001 Elsevier Science Inc. All rights reserved.

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The theory of analytic operator-valued functions is well developed. It seems, however, that relatively little is known about harmonic operator-valued functions. The need for such a theory naturally arises in investigations of the behavior of certain classes of Hilbert space operators under self-adjoint perturbations, where the perturbations depend in a smooth way on a single complex variable. More precisely, because an analytic self-adjoint operator-valued function is necessarily constant, it seems natural to consider harmonic as an appropriate notion of smoothness for such perturbations.

Throughout this paper, let $\mathscr{H}$ be a separable Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the bounded linear operators on $\mathscr{H}$. Denote by $\mathscr{K}(\mathscr{H}) \subseteq \mathscr{L}(\mathscr{H})$ the algebra

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of compact operators on $\mathscr{H}$ and let $\mathscr{K}_{\mathscr{N}\mathscr{N}}(\mathscr{H})$ and $\mathscr{K}_{\mathscr{S}\mathscr{A}}(\mathscr{H})$ be the subalgebras of non-negative compact operators and self-adjoint compact operators, respectively. Recall that the dual of $\mathscr{K}(\mathscr{H})$ is $\mathscr{T}_1$, the trace-class operators in $\mathscr{L}(\mathscr{H})$, where the space $\mathscr{T}_1$ consists of those operators A on $\mathscr{H}$ for which there exists an orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$ of $\mathscr{H}$ such that $\sum_{n=1}^{\infty}\langle A\phi_n, \phi_n\rangle$ is finite. For any $A \in \mathscr{T}_1$, the functional it defines on $\mathscr{K}(\mathscr{H})$ is

$$\tau_A(\cdot) = \mathrm{Tr}(A\cdot) = \sum_{n=1}^{\infty}\langle A(\cdot)\psi_n, \psi_n\rangle,$$

where $\{\psi_n\}_{n=1}^{\infty}$ is (without loss of generality) any choice of orthonormal basis of $\mathscr{H}$ (see [5] for a development of this).

Definition 1. Let $\mathscr{U}$ be an open neighborhood in $\mathbb{C}$. We say $f : \mathscr{U} \to \mathscr{L}(\mathscr{H})$ is *analytic* (respectively, *harmonic*) if for each $\Lambda \in \mathscr{L}(\mathscr{H})^*$ the function $\Lambda \circ f$ is analytic (respectively, harmonic). In particular, we say a function $v : \mathscr{U} \to \mathscr{K}(\mathscr{H})$ is *harmonic* if for each operator $A \in \mathscr{T}_1$, the function $\tau_A \circ v$ is a harmonic function.

A function is analytic in the above sense if and only if it is analytic in the a priori stronger sense of being locally defined by a norm convergent power series of operators (cf. [5]). Because of this, many of the classical results for analytic complex-valued functions easily carry over to analytic operator-valued functions. For example, the Identity Theorem, Cauchy's Theorems, and the Maximum Principle all have natural operator-valued analogs (cf. [1,4] or [3]). Similarly, a function is harmonic in the above sense if and only if it is locally the real part of an analytic operator-valued function and this, in turn, is equivalent to the condition that it is locally the Poisson integral of a strongly continuous operator-valued function. As with analytic operator-valued functions, many of the results for complex-valued harmonic functions carry over directly to the harmonic operator-valued setting. For example, the mean value property, the maximum modulus principle, and the local existence of a harmonic conjugate which is unique up to a constant skew-symmetric operator (i.e., an operator A such that $A^* = -A$) all hold for harmonic operator-valued functions.

In the following lemma we will show that harmonic compact operator-valued functions are characterized by having harmonic diagonal matrix coefficients in any choice of basis. To simplify the proof of this lemma, note that a function $v : \mathscr{U} \to \mathscr{L}(\mathscr{H})$ is harmonic if and only if its real and imaginary parts, $\mathrm{Re}(v) = (1/2)(v + v^*)$ and $\mathrm{Im}(v) = (1/2i)(v - v^*)$, are harmonic. It is, of course, also true that if the positive and negative parts of a harmonic function, $v : \mathscr{U} \to \mathscr{K}_{\mathscr{S}\mathscr{A}}(\mathscr{H})$, are harmonic, then v is harmonic. However, the converse fails even in the case where v takes values in self-adjoint rank one operators. In other words, if a harmonic function v is decomposed as $v = v^+ - v^-$ where $v^+v^- = 0$, then neither v^+ nor v^- needs to be harmonic operators, as is seen by the example $v(x, y) = (x^2 - y^2)\langle\cdot, \phi\rangle\phi$, where $\phi \in \mathscr{H}$. With these remarks, we are ready to establish Lemma 1.

Lemma 1. *Let $v : \mathscr{U} \to \mathscr{K}(\mathscr{H})$, where $\mathscr{U}$ is an open neighborhood in $\mathbb{C}$. The following are equivalent*:
(1) *For each $\phi, \psi \in \mathscr{H}$, the function $\langle v(z)\phi, \psi\rangle$ is harmonic.*
(2) *For each $\phi \in \mathscr{H}$, the function $\langle v(z)\phi, \phi\rangle$ is harmonic.*
(3) *For each operator $A \in \mathscr{T}_1$, the function $\tau_A \circ v$ is harmonic.*
(4) *For some fixed basis $\{\phi_n\}_{n=1}^{\infty}$ of $\mathscr{H}$, each $\langle v(z)\phi_n, \phi_m\rangle$ is harmonic and the norm of $v(z)$ is uniformly bounded on compact subsets of $\mathscr{U}$.*

Proof. By focusing on the real and imaginary parts of v separately, we can assume without loss of generality that v takes values in the self-adjoint compact operators. To see the equivalence of (1) and (2) let $A = \langle \cdot, \phi\rangle\psi$. Then $A^* = \langle \cdot, \psi\rangle\phi$. From the equation

$$\langle \cdot, (\phi + \psi)\rangle(\phi + \psi) - \langle \cdot, \phi\rangle\phi - \langle \cdot, \psi\rangle\psi = \langle \cdot, \phi\rangle\psi + \langle \cdot, \psi\rangle\phi,$$

we see that $\tau_{2\,\mathrm{Re}(A)} \circ v$ is harmonic. A similar calculation (using the vector $\phi + \mathrm{i}\psi$) yields the same conclusion for $\tau_{2\,\mathrm{Im}(A)} \circ v$. Since the real and imaginary parts of A induce harmonic functions, A does as well. Thus, (1) and (2) are equivalent.

Clearly (3) implies (2). To show the converse, we can consider the positive and negative parts of the real and imaginary parts of A separately. Thus, we can assume without loss of generality that A is a non-negative trace class operator. Let

$$A = \sum_{j=0}^{\infty} \lambda_j \langle \cdot, \psi_j\rangle\psi_j,$$

where the λ_j are non-negative and summable and $\{\psi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathscr{H}$. We need to show that

$$\mathrm{Tr}(Av(z)) = \sum_{j=0}^{\infty} \lambda_j \langle v(z)\psi_j, \psi_j\rangle$$

is a harmonic function. Let K be any compact subset of $\mathscr{U}$. Fix $\psi = \psi_j$ for some j and let $W_\psi = \{v(z)\psi \mid z \in K\}$. By the equivalence of (1) and (2) we know that each $\langle v(z)\psi, \phi\rangle$ is harmonic and hence continuous. Thus, W_ψ is a weakly and hence norm bounded subset of $\mathscr{H}$. But then the uniform boundedness principle implies that there exists an $M > 0$ such that $\|v(z)\| \leqslant M$ for all $z \in K$. Hence,

$$|\lambda_j \langle v(z)\psi_j, \psi_j\rangle| \leqslant M|\lambda_j| \quad \forall z \in K$$

is a summable bound and so the Weierstrass M-test implies that the series converges absolutely and uniformly on K. Thus, $\mathrm{Tr}(Av(z))$ is the normal limit of a family of harmonic functions and so it is harmonic.

Finally, suppose that there exists a fixed basis $\{\phi_n\}_{n=1}^{\infty}$ of $\mathscr{H}$ such that each $\langle v(z)\phi_n, \phi_m\rangle$ is harmonic and that the norm of $v(z)$ is uniformly bounded on compact subsets of $\mathscr{U}$. If $\psi = \sum_{n=1}^{\infty} a_n\phi_n$ is any element of $\mathscr{H}$, then the Weierstrass M-test implies that

$$\langle v(z)\psi, \psi\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n \bar{a}_m \langle v(z)\phi_n, \phi_m\rangle$$

is a normal limit of harmonic functions and hence is harmonic. □

In Lemma 1, we actually showed that if a function $v : \mathscr{U} \to \mathscr{L}(\mathscr{H})$ has harmonic matrix coefficients in some fixed choice of basis for $\mathscr{H}$ and $\|v(z)\|$ is a bounded function, then every matrix coefficient of v is harmonic. For a compact operator-valued function this is sufficient to imply that v is a harmonic function. The following example shows that this sufficiency statement is not true for every bounded operator-valued function $v : \mathscr{U} \to \mathscr{L}(\mathscr{H})$.

Example 1. Let $\mathscr{U} = \{z \in \mathbb{C} \mid |z| < 1, \operatorname{Re}(z) > 0\}$ be the right half of the unit disk. For each natural number n, let

$$f_n(z) = z^{\alpha_n},$$

where $\alpha_n = (1/n)^{1/3}$. Define $F : \mathscr{U} \to \mathscr{L}(\mathscr{H})$ to be the infinite dimensional diagonal matrix whose nth diagonal entry is f_n. Each f_n is analytic on $\mathscr{U}$. Moreover, for each $z \in \mathscr{U}$, $\|F(z)\| \leqslant 1$, since each f_n satisfies this bound.

The function F is not an analytic operator-valued function because its derivative does not take values in the bounded operators. Indeed, F' can only be the infinite dimensional diagonal matrix whose nth diagonal entry is

$$f_n' = \alpha_n z^{\alpha_n - 1}.$$

Let ϕ be the square-summable sequence $(1/n)^{\epsilon}$, where $\epsilon = (1+\delta)/2$ for some δ strictly between 0 and 1/3. Then the nth component of $F'(1/2)\phi$ is

$$\left(F'\left(\frac{1}{2}\right)\phi\right)_n = 2^{1-\alpha_n}\left(\frac{1}{n}\right)^{(5+3\delta)/6}.$$

Since the factor $2^{1-\alpha_n}$ is bounded below by 1, this is not a square-summable sequence.

Now consider the function $v = \operatorname{Re}(F)$. The norm of v is uniformly bounded by 1 on $\mathscr{U}$ and so the matrix coefficients of v are harmonic in any choice of basis. But the calculation above shows that partial of v with respect to x does not take values in the bounded operators on $\mathscr{H}$ when $x = 1/2$. Hence, v is not a harmonic operator-valued function.

Now we will use Lemma 1 to establish an operator-valued analog of Harnack's theorem. It should be noted that in [2], Ky Fan proved the following operator-valued analog of Harnack's inequality.

Harnack's inequality (Fan). *Let $\mathscr{H}$ be a complex Hilbert space and let Δ denote the open unit disk in the complex plane. Assume that F is an analytic function on Δ such* *that for each* $z \in \Delta$, $F(z)$ *is a bounded linear operator on* $\mathscr{H}$ *with* $\operatorname{Re}(F(z)) > 0$ (*i.e., non-negative and injective*). *Assume further that* $F(0) = I$. *Then for all* $z \in \Delta$,

$$\frac{1-|z|}{1+|z|} I \leqslant \operatorname{Re}(F(z)) \leqslant \frac{1+|z|}{1-|z|} I.$$

Since we are interested in functions taking values in the compact operator, unless $\mathscr{H}$ is finite dimensional, the above theorem will not apply to our situation. However, an analogous inequality holds for harmonic compact operator-valued functions for similar reasons. We establish this inequality below.

Harnack's inequality. *Let* $\mathscr{U}$ *be an open neighborhood in* $\mathbb{C}$ *and assume that* v: $\mathscr{U} \to \mathscr{K}_{\mathscr{N}\mathscr{N}}(\mathscr{H})$ *is a non-negative operator-valued harmonic function. For any open disk centered at* a *of radius* R, $\Delta(a, R)$, *which is contained in* $\mathscr{U}$ *and any* $a + r\mathrm{e}^{\mathrm{i}\theta} \in \Delta(a, R)$,

$$\frac{R-r}{R+r} v(a) \leqslant v(a + r\mathrm{e}^{\mathrm{i}\theta}) \leqslant \frac{R+r}{R-r} v(a).$$

In other words both

$$v(a + r\mathrm{e}^{\mathrm{i}\theta}) - \frac{R-r}{R+r} v(a) \quad \textit{and} \quad \frac{R+r}{R-r} v(a) - v(a + r\mathrm{e}^{\mathrm{i}\theta})$$

are non-negative operators.

Proof. For each $\phi \in \mathscr{H}$, $\langle v(a + r\mathrm{e}^{\mathrm{i}\theta})\phi, \phi\rangle$ is a real-valued non-negative harmonic function on $\mathscr{U}$. Thus Harnack's inequality implies

$$\frac{R-r}{R+r}\langle v(a)\phi, \phi\rangle \leqslant \langle v(a + r\mathrm{e}^{\mathrm{i}\theta})\phi, \phi\rangle \leqslant \frac{R+r}{R-r}\langle v(a)\phi, \phi\rangle.$$

But this is exactly the content of the theorem. $\square$

The above inequality implies an analog of Harnack's theorem. Recall in the classical setting, Harnack's theorem says that if a sequence v_n of real-valued harmonic functions satisfies $v_k \leqslant v_{k+1}$ for every k, then either v_n converges normally to a harmonic function or $v_n(z) \to \infty$ for all z (cf. [6]).

Harnack's theorem. *Let* $\mathscr{U}$ *be an open neighborhood in* $\mathbb{C}$ *and let* $v_n : \mathscr{U} \to \mathscr{K}_{\mathscr{S}\mathscr{A}}(\mathscr{H})$ *be a non-decreasing sequence of self-adjoint compact operator-valued harmonic functions. Then either there exists a function* $v : \mathscr{U} \to \mathscr{L}(\mathscr{H})$ *such that for each* $\phi \in \mathscr{H}$, *as* $n \to \infty$,

$$\|(v(z) - v_n(z))\phi\| \to 0$$

uniformly in z on compact subsets of $\mathscr{U}$ or for every $z \in \mathscr{U}$, the sequence $v_n(z)$ diverges in the weak operator topology on $\mathscr{L}(\mathscr{H})$. Moreover, if the limit function v exists and takes values in the compact operators, then it is harmonic.

Proof. By subtracting v_1 from each term we can assume without loss of generality that each v_n is non-negative. If there exists a point $z_0 \in \mathscr{U}$ and a vector $\psi \in \mathscr{H}$ such that $\langle v_n(z_0)\psi, \psi\rangle \to \infty$, then the real-valued Harnack's theorem implies that $\langle v_n(z)\psi, \psi\rangle \to \infty$ for each $z \in \mathscr{U}$. Thus, the weak limit of the sequence v_n fails to exist at each point of $\mathscr{U}$.

On the other hand, it is easy to show that for any non-decreasing sequence of non-negative operators, $\{P_n\}_{n=1}^{\infty}$, the sequence either converges in the strong operator topology to a non-negative operator or there exists a $\psi \in \mathscr{H}$ such that $\langle P_n\psi, \psi\rangle \to \infty$ as $n \to \infty$. Thus, if there is no point in $\mathscr{U}$ where v_n is weakly divergent, then for each $z \in \mathscr{U}$, the sequence $v_n(z)$ converges strongly to some bounded operator $v(z)$. Let us show that this convergence is uniform in z on compact subsets of $\mathscr{U}$.

First note that Harnack's inequality implies $\|v(z)\|$ is locally continuous. Indeed, fix an open disk centered at a of radius R, $\Delta(a, R)$, inside $\mathscr{U}$ and let $r < R$. The function v inherits Harnack's inequality on $\Delta(a, R)$ from the sequence v_n. Since v is non-negative, taking the supremum of the expression

$$\frac{R-r}{R+r}\langle v(a)\phi, \phi\rangle \leqslant \langle v(a + r\mathrm{e}^{\mathrm{i}\theta})\phi, \phi\rangle \leqslant \frac{R+r}{R-r}\langle v(a)\phi, \phi\rangle.$$

over all unit vectors ϕ yields

$$\|v(a)\|\frac{R-r}{R+r} \leqslant \|v(a + r\mathrm{e}^{\mathrm{i}\theta})\| \leqslant \frac{R+r}{R-r}\|v(a)\|.$$

Thus, $\|v(a + r\mathrm{e}^{\mathrm{i}\theta})\| \to \|v(a)\|$ as $a + r\mathrm{e}^{\mathrm{i}\theta} \to a$.

Now let K be a compact subset of $\mathscr{U}$ and fix $\phi \in \mathscr{H}$. For each fixed $z \in K$, since $v_n(z)$ is a non-decreasing sequence of non-negative operators converging in the strong operator topology to $v(z)$, $\|v_n(z)\| \leqslant \|v(z)\|$. This, combined with the local continuity of $\|v(z)\|$, implies that there exists a constant $M > 0$ such that $\|v_n(z)\| + \|v(z)\| \leqslant M$ for all $z \in K$. Hence,

$$\begin{aligned}
&\|(v(z) - v_n(z))\phi\|^2 \\
&\quad = \langle (v(z) - v_n(z))^2\phi, \phi\rangle \\
&\quad \leqslant \|(v(z) - v_n(z))\|\langle (v(z) - v_n(z))\phi, \phi\rangle \\
&\qquad \text{(since } v(z) - v_n(z) \text{ is non-negative)} \\
&\quad \leqslant (\|v(z)\| + \|v_n(z)\|)\langle (v(z) - v_n(z))\phi, \phi\rangle \\
&\quad \leqslant M\langle (v(z) - v_n(z))\phi, \phi\rangle.
\end{aligned}$$

By the real-valued Harnack's theorem, the function $\langle v(z) - v_n(z)\phi, \phi\rangle$ converges uniformly to zero on the compact subset K of $\mathscr{U}$. Thus, the strong operator convergence of $v_n(z)$ to $v(z)$ is uniform on compact subsets of $\mathscr{U}$.

Finally, suppose that v_n converges strongly to a limit function v which takes values in the compact operators. Then, for each $\phi \in \mathscr{H}$, $\langle v(z)\phi, \phi\rangle$ is harmonic, since it is a normal limit of the harmonic functions $\langle v_n(z)\phi, \phi\rangle$. Hence, Lemma 1 implies that v is harmonic. $\square$

Example 2. In general, the strong operator limit of a sequence of non-negative compact operator-valued harmonic functions need not be compact operator-valued. For example, let $v_n = I_n$ be the $n \times n$ identity matrix. Then v_n is trivially an increasing sequence of non-negative compact operator-valued functions. The sequence v_n converges in strong operator topology to the identity operator which is not compact unless $\mathscr{H}$ is finite dimensional. Note that this example also shows that even if a a sequence of non-negative compact operator-valued harmonic functions converges strongly, it need not converge in operator norm.

In Example 2, although the limit of the sequence v_n is not compact operator-valued, it is harmonic. The next example shows that the limit need not be harmonic if it is not compact operator-valued.

Example 3. It is possible for an increasing sequence of non-negative compact operator-valued harmonic function to converge to a limit which is not harmonic. Indeed, let $u_n = \mathrm{Re}(f_n) + 1$ where f_n is the analytic function constructed in Example 1. Let w_n be the infinite dimensional diagonal matrix whose ith diagonal element is u_i, if $1 \leqslant i \leqslant n$ and is zero otherwise. The w_n forms an increasing sequence of non-negative compact operator-valued harmonic functions. Clearly, w_n converges in the strong operator topology to $w = v + I$, where $v = \mathrm{Re}(F)$ is the operator-valued function constructed in Example 1. We showed in this example that v is not harmonic. Hence, w is not harmonic.

References

[1] A. Brow, R.G. Douglas, On maximum theorems for analytic operator functions, Acta. Sci. Math. 26 (1965) 325–327.
[2] K. Fan, Inequalities for proper contractions and strictly dissipative operators, Linear Algebra Appl. 105 (1988) 237–248.
[3] P. Hille, Functional Analysis and Semi-groups, Colloquium Publications, AMS, Providence, RI, 1957.
[4] T. Kato, Perturbation Theory for Linear Operators, Springer, Berlin, 1976.
[5] M. Reid, B. Simon, Functional Analysis (Methods of Modern Mathematical Physics), Perspectives in Mathematics, Academic Press, Boston, MA, 1980.
[6] W. Rudin, Real and Complex Analysis, Perspectives in Mathematics, Academic Press, San Diego, 1987.
[7] W. Rudin, Functional Analysis, Perspectives in Mathematics, McGraw-Hill, New York, 1991.
[8] Z. Tao, Analytic operator functions, J. Math. Anal. Appl. 103 (1984) 293–320.