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Singular value decomposition Geršgorin sets

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Abstract

In this note, we introduce the *singular value decomposition Geršgorin set*, $\Gamma^{\mathrm{SV}}(A)$, of an $N \times N$ complex matrix A, where $N \leqslant \infty$. For N finite, the set $\Gamma^{\mathrm{SV}}(A)$ is similar to the standard Geršgorin set, $\Gamma(A)$, in that it is a union of N closed disks in the complex plane and it contains the spectrum, $\sigma(A)$, of A. However, $\Gamma^{\mathrm{SV}}(A)$ is constructed using *column* sums of singular value decomposition matrix coefficients, whereas $\Gamma(A)$ is constructed using *row* sums of the matrix values of A. In the case $N = \infty$, the set $\Gamma^{\mathrm{SV}}(A)$ is defined in terms of the entries of the singular value decomposition of a compact operator A on a separable Hilbert space. Examples are given and applications are indicated.

Keywords: Hilbert space; Compact operator; Singular value decomposition; Geršgorin circle theorem

1. Introduction

The well-known Geršgorin Circle Theorem, stemming from the paper of Geršgorin in 1931 (see [2]), allows one to obtain, by means of easy arithmetic operations on the entries of a given finite-dimensional $N \times N$ complex matrix, a collection of N closed disks, in the complex plane $\mathbb{C}$, whose union is *guaranteed* to include all of the eigenvalues of the given matrix. The beauty and simplicity of Geršgorin's Theorem has undoubtedly inspired much subsequent related research in this area, and many aspects of this topic are covered in the new book [11]. However, extensions of the Geršgorin's circle theorem to infinite dimensions have received far less attention; see however [4,9,10].

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The singular value decomposition of a finite square complex matrix, which produces the associated *singular values* of the given matrix, is now a valued tool for mathematicians and scientists, which has important applications in statistical computations and data compression schemes (which are based on approximating a given matrix by one of lower rank). There are currently robust and well-developed computer programs to carry out this decomposition; for examples, see [1,5]. Similarly, it is the case that there are robust computer programs for numerically determining the *eigenvalues* of a given square complex matrix, where this information is generally vastly different from the information gained from its singular values. Our goal here is to combine these different approaches, thereby producing a Geršgorin-like extension of a singular value decomposition of a compact operator on an infinite dimensional separable Hilbert space.

As background, let $\mathscr{H}$ denote the Hilbert space $\mathbb{C}^N$, where the dimension N can be finite or countably infinite. In the case $N = \infty$, $\mathscr{H}$ is ℓ_2, the vector space of sequences of complex numbers which are absolutely square summable. In this case, we will denote the elements of $\mathscr{H}$ as "infinite length" N-tuples. Define the inner product on elements $x = (x_1, \ldots, x_N)^{\mathrm{T}}$ and $y = (y_1, \ldots, y_N)^{\mathrm{T}}$ in $\mathscr{H}$ by

$$\langle x, y\rangle := \sum_{k=1}^{N} x_k \bar{y} k. \tag{1.1}$$

Let $\|x\| := \sqrt{\langle x, x\rangle} = \left(\sum_{j=1}^{\infty} |x_j|^2\right)^{1/2}$ denote the norm on $\mathscr{H}$, induced by this inner product. This norm induces an operator norm on linear maps, $A : \mathscr{H} \to \mathscr{H}$, namely, $\|A\| := \sup_{\|x\|=1} \|Ax\|$. Let $\mathscr{L}(\mathscr{H})$ denote the set of all linear operators from $\mathscr{H}$ to $\mathscr{H}$ which are bounded with respect to this norm. An operator $B \in \mathscr{L}(\mathscr{H})$ has *finite rank* if and only if there exists a choice of bases on its domain and range spaces $\mathscr{H}$, with respect to which the non-zero entries of B form a finite dimensional matrix. Finally, an operator B is *compact* if and only if it can be expressed as the norm limit of a sequence of finite rank operators. Let $\mathrm{Com}(\mathscr{H})$ denote the set of all compact operators in $\mathscr{L}(\mathscr{H})$. We remark that an operator in $\mathscr{L}(\mathscr{H})$ has a singular value decomposition (defined below) if and only if it is compact.

First, let N be finite, with, say, $N = k$. Then $\mathscr{L}(\mathscr{H})$ is the set of all $k \times k$ complex matrices, i.e., $\mathbb{C}^{k\times k}$. In this case, every element of $\mathscr{L}(\mathscr{H})$ is of finite rank and therefore compact. Hence, our singular value decomposition Geršgorin method, to be defined below, applies to every finite dimensional square complex matrix. Now, let $N = \infty$. Then, $\mathscr{H}$ is a separable infinite dimensional Hilbert space. By the Riesz–Fischer theorem, we can assume, without loss of generality, that $\mathscr{H} = \ell_2$, (see [6, p. 47]). In particular, this gives us a canonical representation of the elements of $\mathscr{H}$ as "infinite length" N-tuples, and hence, the elements of $\mathscr{L}(\mathscr{H})$ are naturally represented as infinite dimensional matrices. In the infinite dimensional case, the compact operators, $\mathrm{Com}(\mathscr{H})$, form a norm closed proper ideal in $\mathscr{L}(\mathscr{H})$.

Let $A \in \mathscr{L}(\mathscr{H})$. By definition, a complex number λ lies in the *spectrum* of A, $\sigma(A)$, if and only if the operator $A - \lambda I$ is non-invertible. In the finite dimensional case, this is equivalent to the existence of a vector $x \in \mathscr{H}$, with $\|x\| > 0$, such that $Ax = \lambda x$. In the infinite dimensional case, the operator $A - \lambda I$ can fail to be invertible, even when no corresponding eigenvector exists. However, this is not true for compact operators. If $A \in \mathrm{Com}(\mathscr{H})$ and $N = \infty$, then the non-zero elements of $\sigma(A)$ are isolated eigenvalues of finite multiplicity and zero is a limit point of eigenvalues of $\sigma(A)$.

The *adjoint* of an operator A is the operator A^* defined by the equation $\langle Ax, y\rangle = \langle x, A^* y\rangle$ for all $x, y \in \mathscr{H}$. As we know, the matrix A^*, which satisfies this equation, is the conjugate transpose of A. An operator A is called *normal* if $AA^* = A^*A$ and *self-adjoint* if $A = A^*$. An operator is

non-negative if $\langle Ax, x\rangle$ is non-negative for all x in $\mathscr{H}$. Clearly, every self-adjoint operator is normal, and a non-negative operator A is necessarily self-adjoint since $\langle Ax, x\rangle = \overline{\langle Ax, x\rangle} = \langle x, Ax\rangle$. The operators which are compact and normal have an important characterization, given by the following Spectral Theorem, [3, p. 183].

Theorem 1.1. *An operator $A \in \mathscr{L}(\mathscr{H})$ is compact and normal if and only if $\mathscr{H}$ has an orthonormal basis of eigenvectors of A, and a corresponding sequence of eigenvalues converges to zero, in modulus, if N is infinite.*

The importance of having an orthonormal basis of eigenvectors is that it allows us to diagonalize a compact and normal operator. More precisely, let $A\gamma_k = \lambda_k\gamma_k$, where $\gamma_k \in \mathscr{H}$ with $\|\gamma_k\| = 1$ and with λ_k a scalar, for $k = 1, 2, \ldots, N$. If U is the unitary matrix whose kth row is $\bar{\gamma}_k^{\mathrm{T}}$ and Λ is the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_N$, then $A = U^*\Lambda U$. Moreover, $U^*U = UU^* = I$, so that

$$UAU^* = \Lambda. \tag{1.2}$$

In other words, when the range and domain of A are given the orthonormal basis $\{\gamma_j\}_{j=1}^N$, the operator A is diagonal. This diagonalization allows us to define functions of the operator A. For example, we define the square root of a non-negative operator A as

$$\sqrt{A} := U^*\sqrt{\Lambda}U, \tag{1.3}$$

where $\sqrt{\Lambda}$ is the diagonal matrix with entries $\sqrt{\lambda_j}$.

Every operator $B \in \mathscr{L}(\mathscr{H})$ has a non-negative *absolute value* operator, defined by $|B| := \sqrt{B^*B}$. The operator B^*B is non-negative since $\langle B^*Bx, x\rangle = \langle Bx, Bx\rangle = \|Bx\|^2 \geqslant 0$ for all $x \in \mathscr{H}$. If B is compact, then B^*B is too, and $\sqrt{B^*B}$ is defined as above. If B^*B is not compact, $\sqrt{B^*B}$ can still be defined, but this requires some sophisticated techniques which will not be necessary for this paper (see [6,8]). The operator $|B|$ is used to define the following decompositions of B.

For any bounded operator B on $\mathscr{H}$, one can define a unique *polar decomposition* of B. A polar decomposition of B is an expression of B as

$$B = Q|B|, \tag{1.4}$$

where Q is a partial isometry on $\mathscr{H}$. That is, Q is an isometry from the closure of the range of $|B|$, denoted by $\overline{\mathrm{Ran}(|B|)}$, to the closure of the range of B, $\overline{\mathrm{Ran}(B)}$. The additional constraint, $\mathrm{Ker}(Q) = \mathrm{Ker}(B)$, uniquely determines Q.

If (and only if) B is compact, it also has a *singular value decomposition* (SVD) which is defined as:

$$B = V^*\Sigma W = \begin{bmatrix} | & \cdots & | \\ \phi & \cdots & \phi_N \\ | & \cdots & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_N \end{bmatrix} \begin{bmatrix} - & \bar{\psi}_1^{\mathrm{T}} & - \\ \vdots & \vdots & \vdots \\ - & \bar{\psi}_N^{\mathrm{T}} & - \end{bmatrix}, \tag{1.5}$$

where V and W are unitary operators on $\mathscr{H}$, and Σ is a diagonal operator on $\mathscr{H}$ whose diagonal entries $\{\sigma_i\}_{i=1}^N$, are a sequence of non-negative, non-increasing numbers are called the *singular values* of B. By definition, $|B|\psi_j = \sigma_j\psi_j$ and $\sigma_j\phi_j = B\psi_j$. The sets $\{\psi_j\}_{j=1}^N$ and $\{\phi_j\}_{j=1}^N$ are orthonormal bases for the Hilbert spaces $\overline{\mathrm{Ran}(|B|)}$ and $\overline{\mathrm{Ran}(B)}$, respectively. The singular value decomposition of an operator B is never unique. In particular, if the collections $\{\phi_j\}_{j=1}^N$, $\{\sigma_j\}_{j=1}^N$ and $\{\psi_j\}_{j=1}^N$ define a singular value decomposition of B, then for any set of complex rotations

$\{e^{i\theta_j}\}_{j=1}^N$, the collections $\{e^{i\theta_j}\phi_j\}_{j=1}^N$, $\{\sigma_j\}_{j=1}^N$ and $\{e^{i\theta_j}\psi_j\}_{j=1}^N$ also define a singular value decomposition of B.

For a compact operator, B, we can calculate its polar decomposition $B = Q|B|$ from a singular value decomposition $B = V^*\Sigma W$. Specifically, by definition, the operator W has rows $\bar{\psi}_j^{\mathrm{T}}$ where $\{\psi_j\}_{j=1}^N$ is an orthonormal basis of eigenvectors of $|B|$. The Spectral Theorem tells us that $|B| = W^*\Sigma W$. Hence, $Q = V^*W$. In particular, this implies the relationship $Q\psi_j = \phi_j$ or, equivalently, $Q^*\phi_j = \psi_j$. We use these identities when it is helpful to express B in terms of a single orthonormal basis.

It will also be useful for us to note that the singular value decomposition allows us to define an inner product notation for compact operators. Specifically, let $B\psi_k = \sigma_k\phi_k$, for $k = 1, \ldots, N$, define a singular value decomposition of a compact operator B. Then, we can express B in inner product terms as

$$B = \sum_{j=1}^{N} \sigma_j \langle \cdot, \psi_j \rangle \phi_j. \tag{1.6}$$

The right hand side is interpreted as follows: For each j, $\sigma_j\langle \cdot, \psi_j\rangle\phi_j$ is the rank one operator on $\mathscr{H}$ which maps $x \in \mathscr{H}$ to the vector $\sigma_j\langle x, \psi_j\rangle\phi_j$, and the sequence of partial sums on the right hand side of (1.6) converges in the operator norm to B.

Definition 1.2. To each $B \in \mathrm{Com}(\mathscr{H})$, we associate the collection, $\mathscr{B}$, of singular value decompositions of B,

$$\mathscr{B} := \left\{ \sum_{l=1}^{N} \sigma_l \langle \cdot, \psi_l \rangle \phi_l : B = \sum_{l=1}^{N} \sigma_l \langle \cdot, \psi_l \rangle \phi_l \right\}, \tag{1.7}$$

where if N is infinite, convergence is in the operator norm. If B is not compact, we set $\mathscr{B} := \emptyset$.

2. Our singular value decomposition Geršgorin set

We are now ready to introduce our *singular value decomposition Geršgorin set*, $\Gamma^{\mathrm{SV}}(B)$, of an $N \times N$ complex matrix B, where $N \leqslant \infty$. When $\mathscr{H}$ is infinite dimensional, an operator has a singular value decomposition if and only if it is compact. In fact, the singular value decomposition of B is an expression of B as a norm limit of finite rank operators. In particular, the SVD Geršgorin method only yields a meaningful eigenvalue inclusion for compact operators.

Definition 2.1. Assume $B \in \mathrm{Com}(\mathscr{H})$ and let $B = V^*\Sigma W = \sum_{l=1}^N \sigma_l\langle \cdot, \psi_l\rangle\phi_l$ be a fixed singular value decomposition of B. We define the SVD matrix values $\{\hat{b}_{\ell,k}\}_{\ell,k=1,\ldots,N}$ of $V^*\Sigma W$ to be

$$\hat{b}_{\ell,k} := \langle B_{\phi\ell,\phi k}\rangle = \sigma_k\langle \phi_\ell, \psi_k\rangle \quad \forall \ell, k = 1, \ldots, N. \tag{2.1}$$

We define the kth SVD column sum of $V^*\Sigma W$ to be $C_k(V^*\Sigma W) := \sum_{\ell=1,\ell\neq k}^N |\hat{b}_{\ell,k}|$, the kth SVD Geršgorin disk to be $\Gamma_k^{\mathrm{SV}}(V^*\Sigma W) := \{z \in \mathbb{C} : |z - \hat{b}_{k,k}| \leqslant C_k(V^*\Sigma W)\}$, and the SVD Geršgorin set of $V^*\Sigma W$ to be $\Gamma_k^{\mathrm{SV}}(V^*\Sigma W) := \cup_{k=1}^N \Gamma_k^{\mathrm{SV}}(V^*\Sigma W)$. Define the SVD Geršgorin set of B to be

$$\Gamma^{\mathrm{SV}}(B) := \cap\{\Gamma^{\mathrm{SV}}(V^*\Sigma W) : V^*\Sigma W \in \mathscr{B}\}. \tag{2.2}$$

If B is a non-compact operator, then $\mathscr{B} = \emptyset$, and we define the SVD Geršgorin set of B, $\Gamma^{\mathrm{SV}}(B)$, to be the entire complex plane $\mathbb{C}$.

We remark that $\Gamma^{\mathrm{SV}}(\mathscr{B})$, defined in (2.2) as the *intersection* of *all* $\Gamma^{\mathrm{SV}}(V^*\Sigma W)$ where $V^*\Sigma W \in \mathscr{B}$, is similar to the finite-dimensional concept of a *minimal Geršgorin set*, treated in [11, Chapter 4]. The intersection property of (2.2) will be used in Theorem 2.3 below.

Given a $B \in \mathrm{Com}(\mathscr{H})$ and given a singular value decomposition $V^*\Sigma W$ of B, where $V^*\Sigma W \in \mathscr{B}$, then an important fact (cf. Theorem 2.2 below) is that $\Gamma^{\mathrm{SV}}(V^*\Sigma W)$ contains the spectrum, $\sigma(B)$, of B. It is interesting to note that the proof of this is essentially the same as the proof for the standard Geršgorin set in the finite dimensional case. (See, for example, [11, p. 4].)

Theorem 2.2. *For any $B \in \mathrm{Com}(\mathscr{H})$ and for any singular value decomposition set $V^*\Sigma W$ from $\mathscr{B}$, then $\sigma(B) \subseteq \Gamma^{\mathrm{SV}}(V^*\Sigma W)$. As this holds for any $V^*\Sigma W$ of $\mathscr{B}$, then (cf. (2.2)) $\sigma(B) \subseteq \Gamma^{\mathrm{SV}}(B)$.*

Proof. The result holds trivially if B is a non-compact operator in $\mathscr{L}(\mathscr{H})$, as $\Gamma^{\mathrm{SV}}(B) = \mathbb{C}$ in this case. Let $B \subseteq \mathrm{Com}(\mathscr{H})$ and fix a singular value decomposition $B := \sum_{l=1}^{N} \sigma_1 \langle \cdot, \psi_l \rangle \phi_l$. Recall from (2.1) that $\hat{b}_{j,k} = \sigma_k \langle \phi_j, \psi_k \rangle$. Suppose $\lambda \in \sigma(B)$. Then, there is an x in $\mathscr{H}$ such that $Bx = \lambda x$ and $\|x\| = 1$. Using Parseval's identity, we calculate

$$Bx = \sum_{l=1}^{N} \sigma_l \langle x, \psi_l \rangle \phi_l = \sum_{l=1}^{N} \sum_{j=1}^{N} \sigma_l \langle x, \phi_j \rangle \langle \phi_j, \psi_l \rangle \phi_j = \sum_{l=1}^{N} \left(\sum_{j=1}^{N} \hat{b}_{j,l} \langle x, \phi_j \rangle \right) \phi_l.$$

On the other hand, $\lambda x = \Sigma_{l=1}^{N} \lambda \langle x, \phi_l \rangle \phi_l$. Since $\|x\|^2 = \Sigma_{l=1}^{N} |\langle x, \phi_l \rangle|^2 = 1$, there exists a k such that $|\langle x, \phi_k \rangle|$ is a non-zero maximal element of the set $\{|\langle x, \phi_l \rangle|\}_{l=1,\ldots,N}$. Fix such a k and equate the coefficients on ϕ_k in the above expansions. This gives us

$$\lambda \langle x, \phi_k \rangle = \sum_{l=1}^{N} \hat{b}_{l,k} \langle x, \phi_l \rangle.$$

This easily implies, by the triangle inequality, that

$$|\lambda - \hat{b}_{k,k}| \leqslant \sum_{l=1, l \neq k}^{N} |\hat{b}_{l,k}| \frac{|\langle x, \phi_l \rangle|}{|\langle x, \phi_k \rangle|} \leqslant \sum_{l=1, l \neq k}^{N} |\hat{b}_{l,j}| = C_k(V^*\Sigma W).$$

By definition, this means λ is in $\Gamma_k^{\mathrm{SV}}(V^*\Sigma W) \subseteq \Gamma^{\mathrm{SV}}(V^*\Sigma W)$ for every singular value decomposition, $V^*\Sigma W$, of B. Thus, $\lambda \in \Gamma^{\mathrm{SV}}(B)$. $\square$

In the special case when B is a compact normal operator, the intersection set $\Gamma^{\mathrm{SV}}(B)$ of (2.2) is just $\sigma(B)$. Moreover, this property characterizes compact normal operators, as we show now.

Theorem 2.3. *Let $B \in \mathscr{L}(\mathscr{H})$. Then, B is a compact normal operator if and only if $\Gamma^{\mathrm{SV}}(B) = \sigma(B)$.*

Proof. If B is not compact, then by Definition 1.2, $\Gamma^{\mathrm{SV}}(B) = \mathbb{C} \neq \sigma(B)$. Let B be a compact operator. Then, B is normal if and only if there exists an orthonormal basis $\{\gamma_j\}_{j=1}^{N}$ in $\mathscr{H}$ such that $B_{\gamma j} = \alpha_j \gamma_j$ and $|\alpha_j| \to 0$ if $\mathscr{H}$ is infinite dimensional.

First, assume that B is normal. Let $\{\alpha_j\}_{j=1}^N = \{|\alpha_j|e^{i\theta_j}\}_{j=1}^N$ be the spectrum of B and assume, without loss of generality, that the sequence $|\alpha_j|$ is non-increasing. Since B is a compact normal operator, the eigenvectors γ_j of B are also eigenvectors of $|B|$. The relations $|B|\gamma_j = |\alpha_j|\gamma_j$ and $B\gamma_j = |\alpha_j|e^{i\theta_j}\gamma_j$ imply that

$$V^*\Sigma W = \sum_{j=1}^{N} |\alpha_j|\langle\cdot, \gamma_j\rangle e^{i\theta_j}\gamma_j$$

is a singular value decomposition of B. In other words, the singular value decomposition matrix coefficients of this SVD of B are $\hat{b}_{j,k} = |\alpha_k|\langle e^{i\theta_j}\gamma_j, \gamma_k\rangle = \alpha_j\delta_{j,k}$ where $\delta_{j,k}$ denotes the Kronecker delta (i.e., 1 if $j = k$ and 0 otherwise). This exactly says that all the SVD Geršgorin column sums of $V^*\Sigma W$ are zero and $\hat{b}_{k,k} = \alpha_k$. Hence, $\Gamma^{SV}(V^*\Sigma W) = \sigma(B)$. This, combined with Theorem 2.2, implies $\Gamma^{SV}(B) = \sigma(B)$.

Conversely, assume $\Gamma^{SV}(B) = \sigma(B)$. Since B is compact, the non-zero spectrum of B consists entirely of isolated eigenvalues of finite multiplicity, which converge to zero in modulus if N is infinite. We show that the equation $\Gamma^{SV}(B) = \sigma(B) = \{\alpha_j\}_{j=1}^N$ implies that B has a singular value decomposition $V^*\Sigma W$ for which all of the SVD Geršgorin column sums of B (i.e., radii of the SVD Geršgorin circles) are zero and the centers of the SVD Geršgorin circles are $\hat{b}_{k,k} = \alpha_k$ for all $k = 1, \ldots, N$. In other words, $\hat{b}_{j,k} = \alpha_j\delta_{j,k}$ for all j and k from 1 to N. Now for every k, $B\phi_k = \sum_{j=1}^N \langle B\phi_k, \phi_j\rangle\phi_j = \sum_{j=1}^N \hat{b}_{k,j}\phi_j = \alpha_k\phi_k$. This establishes that ϕ_k is an orthonormal basis of eigenvectors for B, and the existence of such a basis is a characterization of a normal compact operator.

It remains only to show that $\Gamma^{SV}(B) = \sigma(B)$ implies that B has a singular value decomposition of the form $\sum_{j=1}^N \sigma_j\langle\cdot, e^{i\theta_j}\phi_j\rangle\phi_j$. We show this by induction on M_0, where M_0 is the maximal geometric dimension of the non-zero singular values of B. It is easy to see that if $M_0 = 1$, then every singular value decomposition of B has the desired form. Now assume that B has a decomposition of this form, whenever $M_0 = M - 1$. Without loss of generality, let $B = \sum_{j=1}^l \sigma_j \sum_{k=1}^M \langle\cdot, \psi_k^j\rangle\phi_k^j$ where $l \leqslant N \leqslant \infty$ and $\sigma_1 > \sigma_2 > \cdots > \sigma_l > 0$ are the non-zero eigenvalues of $|B|$, in strictly decreasing order and each of geometric multiplicity M.

Let $\mathscr{A}$ be an indexing set for $\mathscr{B}$, the collection of all singular value decompositions of B. Fix $m \in \mathbb{N}$ sufficiently large so that the closed disks $\overline{\Delta\left(\sigma_j, \frac{3}{m}\right)} := \left\{z \in \mathbb{C} : |z - \sigma_j| \leqslant \frac{3}{m}\right\}$, $j = 1, \ldots, l$, are disjoint. Let

$$K_\alpha := \Gamma^{SV}(B_\alpha)\Big\backslash\left(\cup_{j=1}^l \Delta\left(\sigma_j, \frac{1}{m}\right)\right) \quad (\alpha \in \mathscr{A}).$$

Then, each K_α is compact and

$$\begin{aligned}\cap_{\alpha\in\mathscr{A}} K_\alpha &= \cap_{\alpha\in\mathscr{A}}\left(\Gamma^{SV}(B_\alpha) \cap \left[\cup_{j=1}^l \Delta\left(\sigma_j, \frac{1}{m}\right)\right]^c\right)\\ &= \Gamma^{SV}(B) \cap \left[\left(\cup_{j=1}^l \Delta\left(\sigma_j, \frac{1}{m}\right)\right)\right]^c\\ &= \sigma(B) \cap \left[\cup_{j=1}^l \Delta\left(\sigma_j, \frac{1}{m}\right)\right]^c = \emptyset.\end{aligned}$$

The finite intersection property implies that there exists a finite collection, $B_{\alpha 1}, \ldots, B_{\alpha L}$, such that

$$\cap_{s=1}^{L}\Gamma^{\mathrm{SV}}(B_{\alpha_s}) \subseteq \cup_{j=1}^{l}\Delta\left(\sigma_j, \frac{1}{m}\right).$$

Fix $\sigma \in \{\sigma_1, \ldots, \sigma_l\}$ and, for each $s = 1, \ldots, L$, let $B_s = B_s^{\sigma} = \sum_{j=1}^{M} \sigma\langle\cdot, \psi_j^s\rangle\phi_j^s$ be the restriction of B_{α_s} to the σ eigenspace of $|B|$. For each $s = 1, \ldots, L$, fix a singular value decomposition Geršgorin disk $\overline{\Delta(c_s, \rho_s)}$ of B_s such that $|c_s|$ is maximal, subject to the constraint $\sigma \in \overline{\Delta(c_s, \rho_s)}$. If $|c_s| < \sigma - \frac{3}{m}$ for each $s = 1, \ldots, L$, then each of the disks $\overline{\Delta(c_s, \rho_s)}$ contains the number $\sigma - \frac{2}{m}$, since $\sigma - \frac{2}{m}$ is necessarily closer to c_s than σ is. But this implies that $\sigma - \frac{2}{m}$ lies in $\Gamma^{\mathrm{SV}}(B_s^{\sigma})$ for each $s = 1, \ldots, L$. This contradiction establishes the existence of a SVD Geršgorin disk, inside one of the SVD Geršgorin sets $\Gamma^{\mathrm{SV}}(B_s^{\sigma})$, whose center has modulus at least $\sigma - \frac{3}{m}$. Without loss of generality, assume the center for the first SVD Geršgorin disk of B_1^{σ} has modulus greater than or equal to $\sigma - \frac{3}{m}$. This exactly says that a $\sigma|\langle\phi_1^1, \psi_1^1\rangle| \geqslant \sigma - \frac{3}{m}$. Thus, we have shown that, for every $\sigma \in \{\sigma_1, \ldots, \sigma_1\}$, there exist unit vectors ψ_m^{σ} and ϕ_m^{σ} in the σ eigenspaces of $|B|$ and $|B|^*$, respectively, such that $|\langle\phi_m^{\sigma}, \psi_m^{\sigma}\rangle| \geqslant 1 - \frac{3}{m\sigma}$.

The above argument holds for every natural number greater than m, and taking the limit as $m \to \infty$, establishes the existence of collections $\{\phi^{\sigma j}\}_{j=1}^{l}$ and $\{\psi^{\sigma j}\}_{j=1}^{l}$ of common unit eigenvector of $|B|$ and $|B^*|$ for the eigenvalue σ_j. Hence, for each $j = 1, \ldots, l$, there exists a $\theta_j \in [0, 2\pi)$ such that $\phi^{\sigma_j} := B\psi^{\sigma_j} = \mathrm{e}^{-\mathrm{i}\theta j}\psi^{\sigma j}$.

Now, the operator,

$$B_0 = B - \sum_{j=1}^{l} \sigma_j\langle\cdot, \mathrm{e}^{\mathrm{i}\theta_j}, \phi^{\sigma_j}\rangle\phi^{\sigma_j}$$

satisfies the induction hypothesis. To see this, note that for each m and s in the above argument, the rank one reduction of $B_s^{\sigma} = \sum_{j=1}^{M} \sigma\langle\cdot, \psi_j^s\rangle\phi_j^s$ to $\sum_{j=2}^{M} \sigma\langle\cdot, \psi_j^s\rangle\phi_j^s$ has a SVD Geršgorin set which is smaller than $\Gamma^{\mathrm{SV}}(B_0)$. Hence, $\Gamma^{\mathrm{SV}}(B_0) \subseteq \Gamma^{\mathrm{SV}}(B) = \sigma(B) = \sigma(B_0)$. $\square$

Recall that the singular value decomposition Geršgorin set is defined to be the entire complex plane for non-compact operators, but $\Gamma^{\mathrm{SV}}(B)$ can also be unbonded for a compact operator B. Of course, if B is a finite rank operator, then $\Gamma^{\mathrm{SV}}(B)$ is a bounded subset of $\mathbb{C}$. Hence, in the finite dimensional case, the SVD Geršgorin set is always a bounded subset of $\mathbb{C}$. When B is not of finite rank, the set $\Gamma^{\mathrm{SV}}(B)$ can be unbounded but only if it is the entire complex plane. We show this now.

Theorem 2.4. *Let $N = \infty$ and let $B \in \mathrm{Com}(\mathscr{H})$. Then, for each singular value decomposition $V^*\Sigma W$ of B, the set $\Gamma^{\mathrm{SV}}(V^*\Sigma W)$ is unbounded if and only if $\Gamma^{\mathrm{SV}}(V^*\Sigma W) = \mathbb{C}$. Moreover, this happens if and only if the sequence of SVD column sums, $\{C_k(V^*\Sigma W)\}_{k=1}^{\infty}$, is not bounded. Hence, $\Gamma^{\mathrm{SV}}(B)$ is unbounded if and only if it is the entire complex plane.*

Proof. Let $B = V^*\Sigma W = \sum_{l=1}^{N} \sigma_l\langle\cdot, \psi_l\rangle\phi_l$ be a fixed singular value decomposition of B. Recall, from (2.1), that $\hat{b}_{j,k} = \langle B\phi_j, \phi_k\rangle$, $C_k(B) = \sum_{j\neq k}^{N} |\hat{b}_{j,k}|$, $\Gamma_k^{\mathrm{SV}}(V^*\Sigma W) = \{z \in \mathbb{C} : |z - \hat{b}_{k,k}| \leqslant C_k(B)\}$ and $\Gamma^{\mathrm{SV}}(V^*\Sigma W) = \cup_{k=1}^{N} \Gamma_k^{\mathrm{SV}}(V^*\Sigma W)$. If the sequence $C_k(B)$ is unbounded, by passing to a subsequence, we can assume without loss of generality that $C_k(B) > \|B\|$ for each k, and that $C_k(B) \to \infty$ as $k \to \infty$. Now, $|\hat{b}_{k,k}| = |\langle B\phi_k, \phi_k\rangle| \leqslant \|B\|$, and so, for each k, $R_k := C_k(B) - |\hat{b}_{k,k}|$ is positive. Moreover, the disk $\Delta(0, R_k)$, centered at the origin of radius R_k, is contained in $\Gamma_k^{\mathrm{SV}}(V^*\Sigma W)$. Indeed, if $|z| \leqslant C_k(B) - |\hat{b}_{k,k}|$, then $|z - \hat{b}_{k,k}| \leqslant |z| + |\hat{b}_{k,k}| \leqslant C_k(B)$. Hence,

$\Gamma^{\mathrm{SV}}(V^*\Sigma W) = \mathbb{C}$ when the SVD column sums are not bounded. On the other hand, if the SVD column sums are bounded by some L, then $\Gamma^{\mathrm{SV}}(V^*\Sigma W)$ is contained in the disk centered at the origin of radius $M = L + \|B\|$, since $z \in \Gamma_k^{\mathrm{SV}}(V^*\Sigma W)$ implies that $|z| \leqslant C_k(B) + |\hat{b}_{k,k}| \leqslant L + \|B\|$.

Finally, since the above argument holds for every singular value decomposition of B, the set $\Gamma^{\mathrm{SV}}(B)$ is unbounded if and only if $\Gamma^{\mathrm{SV}}(B) = \mathbb{C}$. $\square$

The proof of the previous theorem also establishes the analogous result for the standard Geršgorin set. Specifically, an operator $B \in \mathscr{L}(\mathscr{H})$ has a Geršgorin set $\Gamma(B)$ which is a proper subset of the complex plane if and only if the sequence $\{R_k(B)\}_{k=1}^N$ of row sums of B is bounded. With this in mind, we define the following two collections of operators in $\mathscr{L}(\mathscr{H})$. Let

$$\mathscr{G} := \{B \in \mathscr{L}(\mathscr{H}) : \Gamma(B) \neq \mathbb{C}\} \quad \text{and} \quad \mathscr{S} := \{B \in \mathscr{L}(\mathscr{H}) : \Gamma^{\mathrm{SV}}(B) \neq \mathbb{C}\}.$$

By construction, $\mathscr{S}$ is a subset of $\mathrm{Com}(\mathscr{H})$. Notice that $\mathscr{G}$ contains the identity operator on $\mathscr{H}$, which is not compact if N is infinite. Thus, $\mathscr{G}$ and $\mathscr{S}$ are distinct subsets of $\mathscr{L}(\mathscr{H})$ when N is infinite. If N is finite, then $\mathscr{S} = \mathscr{G} = \mathbb{C}^{N\times N}$.

We define the equivalence relation $\sim$ on $\mathscr{S}$ by $A \sim B$ if and only if there exists a partial isometry $U : \mathscr{H} \to \mathscr{H}$ such that U^*U is the projection of $\mathscr{H}$ onto $\overline{\mathrm{Ran}(A)}$, that UU^* is the projection of $\mathscr{H}$ onto $\overline{\mathrm{Ran}(B)}$, and that $B = UAU^*$. It is important to note that if $A \sim B$, then $\Gamma^{\mathrm{SV}}(A) = \Gamma^{\mathrm{SV}}(B)$. Indeed, if $B = UAU^*$ and $A = \Sigma_{j=1}^N \sigma_j \langle\cdot, \psi_j\rangle\phi_j$, then $B = \Sigma_{j=1}^N \sigma_j \langle\cdot, U\psi_j\rangle U\phi_j$ and

$$\hat{b}_{j,k} = \langle BU\phi_j, U\phi_k\rangle = \langle A\phi_j, \phi_k\rangle = \hat{a}_{j,k}$$

for all $j, k = 1, 2, \ldots, N$. Thus, the singular value decomposition Geršgorin set is independent of the equivalence class representative in $\mathscr{S}_0 = \mathscr{S}/\sim$, where $\mathscr{S}/\sim$ denotes $\mathscr{S}$ modulo the given equivalence relation. We will denote the singular value decomposition Geršgorin set of an element $[A]$ of $\mathscr{S}_0$ by $\Gamma^{\mathrm{SV}}([A])$.

Notice that if $A = V^*\Sigma W$, then $A \sim \Sigma W V^*$. Hence, each equivalence class in $\mathscr{S}_0$ has a representative of the form ΣU where Σ is a positive diagonal operator and U is a partial isometry. Moreover, each equivalence class has a unique representative of this form since, if $\Sigma_1 U_1 = \Sigma_2 U_2$, then $\Sigma_2^{-1}\Sigma_1 = U_2 U_1^*$ is unitary and a positive diagonal operator. Hence, it must be the identity operator on Ran (U_1) = Ran (U_2) and so $\Sigma_1 = \Sigma_2$ and $U_1 = U_2$. This observation shows that there is a well-defined map Ψ from $\mathscr{S}_0$ to $\mathscr{L}(\mathscr{H})$, given by $\Psi([V^*\Sigma W]) = \Sigma W V^*$.

Theorem 2.5. *There is a natural bisection*

$$\Psi : \mathscr{S}_0 \to \mathscr{G}_0$$

between $\mathscr{S}_0 = \mathscr{S}/\sim$, equipped with the singular value decomposition Geršgorin method, and, $\mathscr{G}_0$ a subset of $\mathscr{G} \cap \mathrm{Com}(\mathscr{H})$, equipped with the standard Geršgorin method.

Proof. We have shown that $\Psi([V^*\Sigma W]) = \Sigma W V^*$ is a well-defined function on $\mathscr{S}_0$. Moreover, Ψ takes values in $\mathscr{G}$. To see this, let $[A]$ denote an equivalence class in $\mathscr{S}_0$ and let $A = V^*\Sigma W$ be a singular value decomposition of A with $\Gamma^{\mathrm{SV}}(V^*\Sigma W) \neq \mathbb{C}$. Then $A \sim VAV^*$. Since the leading factor of $B = VAV^*$ is the identity operator on the closure of the range of B, the singular value decomposition matrix coefficients of B are

$$\hat{b}_{j,k} = \langle Be_j, e_k\rangle = b_{k,j}. \tag{2.3}$$

It follows easily that $\Gamma^{\mathrm{SV}}(V^*\Sigma W) = \Gamma^{\mathrm{SV}}(\Sigma W V^*) = \Gamma(\Sigma W V^*) \neq \mathbb{C}$.

An argument similar to the well-definedness calculation for Ψ, establishes that Ψ is one-to-one on $\mathscr{S}_0$. Let $\mathscr{G}_0 = \Psi(\mathscr{S}_0)$. Then, Ψ is a bijection from $\mathscr{S}_0$ to $\mathscr{G}_0$. We have shown that Ψ maps into $\mathscr{G}$ and, since the compact operators are an ideal in $\mathscr{L}(\mathscr{H})$, the image of $\mathscr{S}_0$ under Ψ lies in Com $(\mathscr{H})$. $\square$

Even when N is finite, $\mathscr{G}_0$ is a proper subset of $\mathscr{G} \cap$ Com $(\mathscr{H})$, since each element of the range of Ψ has a factorization of the form ΣU where Σ is a positive, diagonal operator and U is a partial isometry. In fact, $B \in \mathscr{G}_0$ if and only if B is a compact operator, with a bounded Geršgorin set and B has factorization of the form ΣU. However, the bijection $\Psi : \mathscr{S}_0 \to \mathscr{G}_0$ gives the relationship between the SVD Geršgorin set and the standard Geršgorin set for operators in $\mathscr{G}_0$.

Theorem 2.6. *For each $B \in \mathscr{G}_0$, the singular value decomposition Geršgorin set of B, $\Gamma^{\mathrm{SV}}(B)$, is contained in the Geršgorin set of B, $\Gamma(B)$. That is,*

$$\forall B \in \mathscr{G}_0,\ \Gamma^{\mathrm{SV}}(B) \subseteq \Gamma(B).$$

Proof. Let $B \in \mathscr{G}_0 = \Psi(\mathscr{S}_0)$, with $B = \Sigma U$. Then,

$$\Gamma^{\mathrm{SV}}(B) \subseteq \Gamma^{\mathrm{SV}}(\Sigma U) = \Gamma(\Sigma U) = \Gamma(B). \qquad \square \tag{2.4}$$

3. Examples and applications

Example 1. Let $\mathscr{H} = \ell_2$ be the sequence space of absolutely square summable complex sequences. Let a_j be a sequence of real numbers with $\sum_{j=1}^{\infty} |a_j| < \infty$, and define the operator $A : \mathscr{H} \to \mathscr{H}$ by the infinite matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \dots \\ a_2 & a_3 & a_4 & \dots \\ a_3 & a_4 & a_5 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3.1}$$

It is known that A is a compact self-adjoint operator (see [3, p. 101]), whose singular value decomposition Geršgorin set exactly equals the spectrum of A. The standard Geršgorin set of A is bounded but does not generally equal the spectrum of A. Specifically,

$$\Gamma^{\mathrm{SV}}(A) = \sigma(A) \subseteq \Gamma(A) = \bigcup_{k=1}^{\infty} \left\{ z \in \mathbb{C} : |z - a_{2k-1}| \leqslant \sum_{j \geqslant k,\, j \neq 2k-1} |a_j| \right\}. \tag{3.2}$$

As a specific example of the above, let the matrix A of (3.1), be defined by

$$a_j = \frac{1}{2j} \quad (j = 1, 2, \dots). \tag{3.3}$$

In this case, it can be verified that if A_n is the $n \times n$ leading principal submatrix of A, then the spectra of these principal submatrices are given by

$$\sigma(A_1) = \{0, 5\},$$
$$\sigma(A_2) = \{0.625; 0\},$$

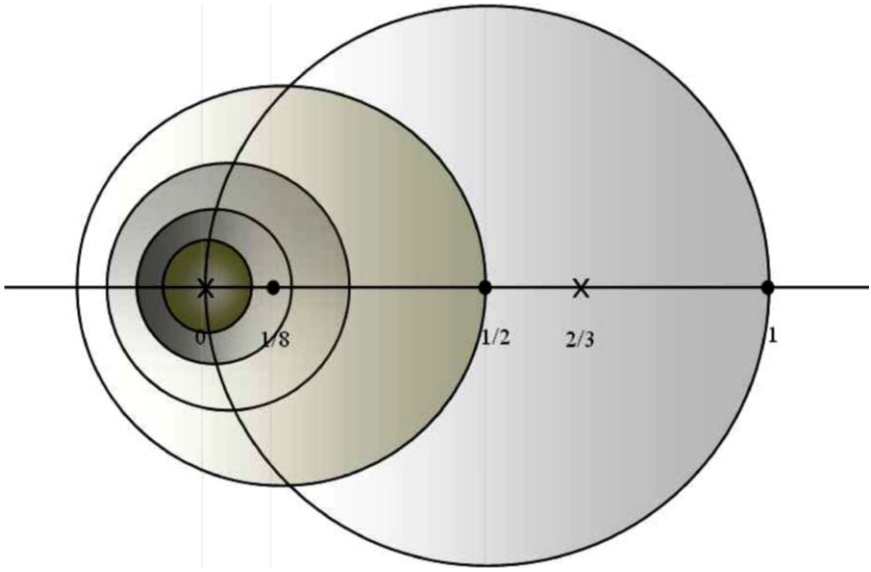

Fig. 3.1. $\Gamma(A)$ (shaded region), $\Gamma^{\mathrm{SV}}(A) = \sigma(A)$ ("x's").

$$\sigma(A_3) = \{0.65625; 0, 0\},$$
$$\sigma(A_4) = \{0.6640625; 0, 0, 0\},$$
$$\sigma(A_5) = \{0.666015625; 0, 0, 0, 0\},$$
$$\sigma(A_6) = \{0.66650390625; 0, 0, 0, 0, 0\}, \text{ and finally}$$
$$\sigma(A_\infty) = \left\{\frac{2}{3}; 0, 0, 0, \ldots\right\}.$$

For the choice of (3.3) for the matrix of (3.1) its associated sets $\sigma(A) = \Gamma^{\mathrm{SV}}(A)$ and $\Gamma(A)$, are shown in Fig. 3.1. (We note that $\Gamma(A)$, the union of an *infinite* number of closed disks, reduces to the union of first two closed disks.)

Example 2. Consider the normal matrix

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.4}$$

Clearly, $\sigma(B) = \{-1, 1\}$ and $|B| := \sqrt{B^*B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Consider the following two singular value decompositions of B:

$$B = V_1^* \Sigma_1 W_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.5}$$

and

$$B = V_2^* \Sigma_2 W_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{3.6}$$

A routine calculation shows that

$$\Gamma^{\mathrm{SV}}(V_1^*\Sigma_1 W_1) = \Gamma(\Sigma_1 W_1 V_1^*) = \Gamma\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \{-1\} \cup \{1\},$$

whereas,

$$\Gamma^{\mathrm{SV}}(V_2^*\Sigma_2 W_2) = \Gamma(\Sigma_2 W_2 V_2^*) = \Gamma\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \{z \in \mathbb{C} : |z| \leqslant 1\}.$$

This illustrates that even for a normal operator B, the SVD Geršgorin set $\Gamma^{\mathrm{SV}}(V^*\Sigma W)$, of some fixed SVD of B, can *fail* to give exactly the eigenvalues of this operator B, while both singular value decompositions of B above give necessarily the same singular values for B. (Curiously, it does not seem possible for MATLAB users to ask if there are *other* singular value decompositions for their given matrix, as one gives the matrix to MATLAB, and *a* singular value decomposition is delivered!)

In general, we have shown here that one can attain singular value decomposition Geršgorin sets for any operator $B \in \mathrm{Com}(\mathscr{H})$, which gives added information about the spectra of such operators. This can easily be extended to Brauer ovals of Cassini as well, but this is left for a future work.

We finally wish to mention that, once having calculated the singular value decomposition of an operator, getting eigenvalue estimates of this operator is computationally easy, using our SVD Geršgorin method. More precisely, our method allows one to use a given singular value decomposition of an operator to cheaply obtain eigenvalue estimates which are generally different from the standard Geršgorin estimates.

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