A Note on an Open Question on ω- and τ-Matrices

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ABSTRACT

In a recent paper by Engel and Schneider, it was asked if, for every $n \geqslant 1$, $A \in \tau_{\langle n \rangle}$ implies $(A+D) \in \tau_{\langle n \rangle}$ for every $D = \mathrm{diag}[d_1, d_2, \ldots, d_n]$ with $d_i \geqslant 0$, $1 \leqslant i \leqslant n$. We answer this question in the *negative*. More precisely, we show that for, any $n \geqslant 3$, the set $\mathcal{Q}(\tau_{\langle n \rangle}) := \{D \in \mathbf{C}^{n,n} : (A+D) \in \tau_{\langle n \rangle} \text{ for all } A \in \tau_{\langle n \rangle}\}$ is exactly given by $\mathcal{Q}(\tau_{\langle n \rangle}) = \{\gamma I_n : \gamma \geqslant 0\}$.

1. INTRODUCTION

In a recent paper, Engel and Schneider [1] have introduced two new and important classes of matrices in $\mathbf{C}^{n,n}$, called *ω-matrices* and *τ-matrices*, and they have established some interesting properties for these matrices. These matrices are of considerable interest, since they include, as special cases, Hermitian matrices, M-matrices, and totally nonnegative matrices.

One of the open questions raised in [1, 7.5 (iv)] is this. Given an arbitrary τ-matrix A in $\mathbf{C}^{n,n}$, is $A + D$ again an τ-matrix for every nonnegative diagonal matrix in $\mathbf{C}^{n,n}$? Theorem 1, stated below in this section, but proved in Sec. 2, shows that this is in general false for $n \geqslant 3$. The remainder of this section is devoted to introducing necessary notation.

Following Engel and Schneider [1], let $\langle n \rangle := \{1, 2, \ldots, n\}$ for every positive integer n, let α denote a subset of $\langle n \rangle$, and let $|\alpha|$ denote the cardinality of α. Next, for $A = [a_{i,j}] \in \mathbf{C}^{n,n}$, $A[\alpha]$ denotes the principal sub-

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matrix of A determined by α, i.e., $A[\alpha]=[a_{i,j}]$, where $i,j\in\alpha$; $\operatorname{spec}(A)$ denotes the set of eigenvalues of A, and $\det A$ denotes the determinant of A. Further, I_n denotes the identity matrix in $\mathbf{C}^{n,n}$ and $\mathfrak{D}_n$ and $\mathfrak{D}_n^+$ are, respectively, the collection of all real diagonal matrices and that of all nonnegative diagonal matrices in $\mathbf{C}^{n,n}$.

Next, set

$$l(A):=\min\{\lambda:\lambda\in\operatorname{spec}(A)\cap\mathbf{R}\} \tag{1.1}$$

for any $A\in\mathbf{C}^{n,n}$, with the usual convention that $l(A):=+\infty$ if A has no real eigenvalues. Then (cf. Engel and Schneider [1]), a matrix $A\in\mathbf{C}^{n,n}$ is called an *ω-matrix* if the following are satisfied:

$$l(A[\alpha])<+\infty \qquad \text{for all } \alpha \text{ with} \quad \varnothing\neq\alpha\subseteq\langle n\rangle, \tag{1.2}$$

and

$$\varnothing\neq\alpha\subseteq\beta\subseteq\langle n\rangle \quad \text{implies} \quad l(A[\beta])\leqslant l(A[\alpha]), \tag{1.3}$$

this latter condition being called *eigenvalue monotonicity*. The collection of all ω-matrices in $\mathbf{C}^{n,n}$ is then denoted by $\omega_{\langle n\rangle}$. Obviously,

$$A\in\omega_{\langle n\rangle} \quad \text{iff} \quad A[\alpha]\in\omega_{\langle|\alpha|\rangle} \quad \text{for all } \varnothing\neq\alpha\subseteq\langle n\rangle. \tag{1.4}$$

A matrix $A\in\omega_{\langle n\rangle}$ is further called a *τ-matrix* in [1] if, in addition,

$$l(A)\geqslant 0, \tag{1.5}$$

and the collection of all τ-matrices in $\mathbf{C}^{n,n}$ is then denoted by $\tau_{\langle n\rangle}$.

If $\mathbb{S}$ denotes an arbitrary set in $\mathbf{C}^{n,n}$, we define

$$\mathcal{Q}(\mathbb{S}):=\{D\in\mathbf{C}^{n,n}:(A+D)\in\mathbb{S}\ \forall A\in\mathbb{S}\}. \tag{1.6}$$

For example, if $\mathcal{H}_n$ is the set of all Hermitian matrices in $\mathbf{C}^{n,n}$, then obviously $\mathcal{Q}(\mathcal{H}_n)=\mathcal{H}_n$. Similarly, if $\mathcal{H}_n^+$ is the set of all Hermitian positive semidefinite matrices in $\mathbf{C}^{n,n}$, then $\mathcal{Q}(\mathcal{H}_n^+)=\mathcal{H}_n^+$.

Our main result is then a characterization of $\mathcal{Q}(\omega_{\langle n\rangle})$ and $\mathcal{Q}(\tau_{\langle n\rangle})$, which we state as

THEOREM 1. *For any* $n \geqslant 3$,

$$\mathcal{C}(\omega_{\langle n \rangle}) = \{\gamma I_n : \gamma \text{ real}\}, \tag{1.7}$$

and

$$\mathcal{C}(\tau_{\langle n \rangle}) = \{\gamma I_n : \gamma \geqslant 0\}, \tag{1.8}$$

while for $1 \leqslant n \leqslant 2$,

$$\mathcal{C}(\omega_{\langle n \rangle}) = \mathfrak{D}_n, \tag{1.9}$$

and

$$\mathcal{C}(\tau_{\langle n \rangle}) = \mathfrak{D}_n^+. \tag{1.10}$$

2. PROOFS

We begin by stating

LEMMA 1. *For* $A = [a_{i,j}] \in \mathbf{C}^{2,2}$ *to be an element of* $\omega_{\langle 2 \rangle}$, *it is necessary and sufficient that*

$$\begin{aligned} &a_{i,i} \quad \text{are real}, \qquad i = 1,2, \\ &a_{1,2}a_{2,1} \geqslant 0. \end{aligned} \tag{2.1}$$

Similarly, for $A = [a_{i,j}] \in \mathbf{C}^{2,2}$ *to be an element of* $\tau_{\langle 2 \rangle}$, *it is necessary and sufficient that*

$$\begin{aligned} &a_{i,i} \geqslant 0, \qquad i = 1,2, \\ &a_{1,2}a_{2,1} \geqslant 0, \\ &\det A \geqslant 0. \end{aligned} \tag{2.2}$$

We remark that the sufficiency of (2.1) follows as a special case of Engel and Schneider [1, Theorem 6.4], while the necessity of (2.1) is a direct consequence of (1.2)–(1.3), which can also be deduced from [1, Corollary 4.6]. The remainder follows similarly.

LEMMA 2. *For any $n \geqslant 1$, $D \in \mathcal{C}(\omega_{\langle n \rangle})$ or $D \in \mathcal{C}(\tau_{\langle n \rangle})$ implies D is a real diagonal matrix.*

Proof. Writing $D=[d_{i,j}] \in \mathbf{C}^{n,n}$, it is clear from (1.2) that the diagonal entries of any ω-matrix are real, whence the hypothesis $D \in \mathcal{C}(\omega_{\langle n \rangle})$ implies $d_{i,i}$ is real for all $i \in \langle n \rangle$. Next, for $n \geqslant 2$, suppose that $|d_{i,j}|>0$ for some $i \neq j$; $i,j \in \langle n \rangle$. We may suppose, in fact, that $i=1$, $j=2$, and we write $0 \neq d_{1,2} = |d_{1,2}|e^{i\theta}$. Now, the particular triangular matrix $A(r):=[a_{i,j}(r)] \in \mathbf{C}^{n,n}$, defined by

$$a_{2,1} := -re^{-i\theta}, \qquad r \geqslant 0;$$

$$a_{i,j} := 0 \text{ for all other } i,j \in \langle n \rangle, \tag{2.3}$$

is in $\tau_{\langle n \rangle}$ for *every* choice of $r \geqslant 0$. But the hypothesis $D \in \mathcal{C}(\omega_{\langle n \rangle})$ implies $A(r)+D$ is in $\omega_{\langle n \rangle}$ for all $r \geqslant 0$, which implies [cf. (1.4)] that $(A(r)+D)[1,2] \in \omega_{\langle 2 \rangle}$. Hence, on applying the second part of (2.1) of Lemma 2.1, we must have

$$\left(-re^{-i\theta}+d_{2,1}\right)\left(|d_{1,2}|e^{i\theta}\right) \geqslant 0 \qquad \text{for all} \quad r \geqslant 0,$$

which is false for all r sufficiently large. Thus, D is a real diagonal matrix. The proof for $D \in \mathcal{C}(\tau_{\langle n \rangle})$ is exactly the same. ∎

We remark that because the matrix $A(r)$, defined in (2.3), for $\theta=0$ is also a singular M-matrix, the method of proof of Lemma 2 can also be used to establish

$$\mathcal{C}(\mathfrak{M}_n)=\mathfrak{D}_n^+ \qquad \text{for all} \quad n \geqslant 1, \tag{2.4}$$

where $\mathfrak{M}_n$ denotes the collection of all (possibly singular) M-matrices in $\mathbf{C}^{n,n}$ (i.e., $A=[a_{i,j}] \in \mathfrak{M}_n$ implies that A is real, $a_{i,j} \leqslant 0$ for all $i \neq j$, and $A+\operatorname{diag}[d_1,d_2,\ldots,d_n]$ is nonsingular for any $\mathbf{d}=[d_1,d_2,\ldots,d_n]^T$ with positive components). See also Fiedler and Pták [2] and Willson [3].

Proof of Theorem 1. We first note that the case $n=1$ of Theorem 1 is completely trivial, while for $n=2$, (1.9) and (1.10) of Theorem 1 follow directly from Lemmas 1 and 2. Thus, assume $n \geqslant 3$. Now, for any real number γ, it is evident from (1.2) and (1.3) that $(A+\gamma I_n) \in \omega_{\langle n \rangle}$ for all

$A\in\omega_{\langle n\rangle}$, whence by definition (1.6),

$$\mathcal{Q}(\omega_{\langle n\rangle})\supset\{\gamma I_n:\gamma \text{ real}\}.$$

To establish the reverse inclusion above when $n\geqslant 3$, consider the following matrix in $\mathbf{C}^{3,3}$:

$$B:=\begin{bmatrix}10 & 10 & 10\\ 10 & 10 & 10\\ 8 & 3 & 8\end{bmatrix}. \tag{2.5}$$

If we write $l[\mu]:=l(B[\mu])$ for any nonempty subset $\mu\subseteq\langle 3\rangle$, it can be verified [cf. (1.1)] that

$$\begin{aligned}&l[1]=l[2]=10; \qquad l[3]=8;\\ &l[1,2]=l[1,3]=0; \qquad l[2,3]=9-\sqrt{31}\doteqdot 3.432236;\\ &l[1,2,3]=0.\end{aligned} \tag{2.6}$$

Hence, by definition, $B\in\tau_{\langle 3\rangle}$. Similarly, it is readily verified that if

$$\tilde{B}:=\begin{cases}B, & n=3,\\ B\oplus\mathcal{O}_{n-3}, & n>3,\end{cases} \tag{2.7}$$

then $\tilde{B}\in\tau_{\langle n\rangle}$ for any $n\geqslant 3$, where $\mathcal{O}_i$ represents the null matrix in $\mathbf{C}^{i,i}$.

Returning to (2.5), consider $B+\text{diag}[1,d,0]$, where d is any real number satisfying $0\leqslant d\leqslant 1$, i.e.,

$$B+\text{diag}[1,d,0]=\begin{bmatrix}11 & 10 & 10\\ 10 & 10+d & 10\\ 8 & 3 & 8\end{bmatrix}=:\hat{B}(d). \tag{2.8}$$

Now, $\hat{B}(d)$ is *not* an element of $\omega_{\langle 3\rangle}$ for *any* $0\leqslant d\leqslant 1$. To indicate this, on setting $\hat{l}_d[\mu]:=l(\hat{B}(d)[\mu])$, we have from (2.8) that

$$\hat{l}_d[1,3]=\frac{19-\sqrt{329}}{2}\doteqdot 0.430821, \qquad \text{independent of } d. \tag{2.9}$$

On the other hand, $\hat{l}_d[1,2,3]$, the minimum real eigenvalue of $\hat{B}(d)$ [which always exists, since $\hat{B}(d)$ is real and of odd order], is a discontinuous function of d on $[0,1]$, with $\hat{B}(d)$ having one real eigenvalue (the Perron root) for $0 \leqslant d < d^* \doteq 0.533575$, while for $d^* \leqslant d \leqslant 1$, $\hat{B}(d)$ has three real eigenvalues. After some tedious calculations, which we omit, on zeros of cubic polynomials, it can be shown that

$$\hat{l}_d[1,2,3] > \hat{l}_d[1,3] \qquad \text{for all} \quad 0 \leqslant d \leqslant 1. \tag{2.10}$$

As this violates the monotonicity property of (1.3), we have $\hat{B}(d) \notin \omega_{\langle 3 \rangle}$ for all $d \in [0,1]$. We give in Table 1 the discontinuous values of $\hat{l}_d[1,2,3]$ for $d = 0(0.1)1$, along with $\hat{l}_d[1,3]$ of (2.9).

Continuing the proof of Theorem 1, consider any $D = [d_{i,j}] \in \mathcal{Q}(\omega_{\langle n \rangle})$ for $n \geqslant 3$. From Lemma 2, D is a real diagonal matrix in $\mathbf{C}^{n,n}$, and we thus write $D := \operatorname{diag}[d_1, d_2, \ldots, d_n]$ and set

$$\sigma_1 := \min\{d_i : i \in \langle n \rangle\}; \qquad \sigma_2 := \max\{d_1 : i \in \langle n \rangle\}.$$

Since $(A + \sigma_1 I_n) \in \omega_{\langle n \rangle}$ iff $A \in \omega_{\langle n \rangle}$, we may assume without loss of generality that $\sigma_1 = 0$ and that $\sigma_2 \geqslant 0$. If $\sigma_2 > 0$, select any subset μ of $\langle n \rangle$ with $|\mu| = 3$ such that

$$0 = \min\{d_i : i \in \mu\}; \qquad \max\{d_i : i \in \mu\} = \sigma_2 > 0.$$

Again without loss of generality, we may assume that $\mu = \{1,2,3\}$, and that

$$d_1 = \sigma_2 > 0; \qquad 0 \leqslant d_2 \leqslant \sigma_2; \qquad d_3 = 0.$$

Now, the hypothesis $D \in \mathcal{Q}(\omega_{\langle n \rangle})$ in particular implies that $(D + \sigma_2 \tilde{B}) \in \omega_{\langle n \rangle}$, where $\hat{B}$, an element of $\tau_{\langle n \rangle}$, is defined in (2.7). Using (1.4), we must have that $(D + \sigma_2 \tilde{B})[\{1,2,3\}] \in \omega_{\langle 3 \rangle}$, or equivalently, that [cf. (2.7) and (2.8)]

$$\operatorname{diag}[\sigma_2, d_2, 0] + \sigma_2 B = \sigma_2 \left\{ B + \operatorname{diag}\left[1, \frac{d_2}{\sigma_2}, 0\right] \right\} = \sigma_2 \hat{B}\left(\frac{d_2}{\sigma_2}\right)$$

TABLE 1

d	$\hat{l}_d[1,2,3]$	$\hat{l}_d[1,3]$	d	$\hat{l}_d[1,2,3]$	$\hat{l}_d[1,3]$
0	26.505681	0.430821	0.5	26.670035	0.430821
0.1	26.538191	0.430821	0.6	1.235369	0.430821
0.2	26.570880	0.430821	0.7	1.141984	0.430821
0.3	26.603750	0.430821	0.8	1.081903	0.430821
0.4	26.636802	0.430821	0.9	1.036608	0.430821
			1.0	1.000000	0.430821

is an element of $\omega_{\langle 3\rangle}$, where $0 \leqslant d_2/\sigma_2 \leqslant 1$. But, our previous discussion in fact shows that $\sigma_2 \hat{B}(d_2/\sigma_2)$ is *not* in $\omega_{\langle 3\rangle}$, a contradiction, whence $\sigma_2 = 0$. Thus, $\sigma_1 = \sigma_2$, so that $D = \sigma_1 I_n$, and

$$\mathcal{E}(\omega_{\langle n\rangle}) = \{\gamma I_n : \gamma \text{ real}\} \qquad \text{for all } n \geqslant 3,$$

which establishes (1.7) of Theorem 1. Next, because $\tilde{B}$ of (2.7) is an element of $\tau_{\langle n\rangle}$, the above method similarly shows that

$$\mathcal{E}(\tau_{\langle n\rangle}) = \{\gamma I_n : \gamma \geqslant 0\} \qquad \text{for all} \quad n \geqslant 3,$$

which establishes (1.8) of Theorem 1. ■

3. REMARKS AND OPEN QUESTIONS

One consequence of Theorem 1 is that $\tau_{\langle n\rangle}$ is substantially different from $\mathfrak{M}_n$ in that [cf. (2.4)] $\mathcal{E}(\mathfrak{M}_n) = \mathfrak{D}_n^+$ for all $n \geqslant 1$, while $\mathcal{E}(\tau_{\langle n\rangle}) = \{\gamma I_n : \gamma \geqslant 0\}$. This means that the property of being an ω-matrix or a τ-matrix is *not* determined solely by the cycles of the matrix (cf. [1, 7.5(i)]). It would, however, be interesting to characterize those subsets, $\tau^d_{\langle n\rangle}$, of $\tau_{\langle n\rangle}$ for which

$$\mathcal{E}(\tau^d_{\langle n\rangle}) = \mathfrak{D}_n^+. \tag{3.1}$$

For example, it is not difficult to verify, using (2.4), that $\mathfrak{M}_n \cup \mathcal{H}_n^+$ is a subset of $\tau_{\langle n\rangle}$ such that

$$\mathcal{E}(\mathfrak{M}_n \cup \mathcal{H}_n^+) = \mathfrak{D}_n^+. \tag{3.2}$$

An open question is if there exists a *maximal* subset $\tau^d_{\langle n\rangle}$ satisfying (3.1).

Another open question is this. If $A \in \mathbf{C}^{n,n}$ satisfies the Hadamard inequality, i.e.,

$$\forall \alpha, \beta \subseteq \langle n\rangle \qquad \text{with} \quad \alpha \cap \beta = \varnothing,$$

$$0 < \det A[\alpha \cup \beta] \leqslant \det A[\alpha] \det A[\beta], \tag{3.3}$$

then does there exist a real diagonal matrix $D \in \mathbf{C}^{n,n}$ such that

$$(A + D) \in \tau_{\langle n\rangle}? \tag{3.4}$$

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