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Dedicated to Ky Fan, on the occasion of his sixty-second birthday, September 19, 1976

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If $G(C)$ denotes the minimal Geršgorin set for $C \in \mathbb{C}^{n,n}$, and if, for any nonempty subset α of the first n positive integers, $C[\alpha]$ denotes the principal minor of C determined by α, then conditions are determined which characterize matrices A and B in $\mathbb{C}^{n,n}$ such that the inclusions $G((D+B)[\alpha]) \subseteq G((D+A)[\alpha])$ are valid for all subsets α of the first n positive integers, and for all diagonal matrices D in $\mathbb{C}^{n,n}$. Connections with the newly defined set of ω-matrices are also included.

1. INTRODUCTION

For any matrix $A = [a_{i,j}] \in \mathbb{C}^{n,n}$, let $G(A)$ denote its *minimal Geršgorin set* (cf. [6]), i.e.,

$$G(A) := \bigcap_{x \in \mathbb{R}_+^n} \left\{ \bigcup_{i=1}^{n} [z \in \mathbb{C} : |z - a_{i,i}| \leqslant \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i,j}| x_j / x_i] \right\}, \tag{1.1}$$

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where $\mathbb{R}^n_+$ denotes the collection of column vectors $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ in $\mathbb{R}^n$ with $x_i > 0$ for all $1 \leqslant i \leqslant n$, and where the sum in (1.1) is defined to be zero when $n = 1$. With this notation, we seek conditions on matrices $A = [a_{i,j}]$ and $B = [b_{j,i}]$ in $\mathbb{C}^{n,n}$ which insure that

$$\begin{cases} G((D + B)[\alpha]) \subseteq G((D + A)[\alpha]) \quad \text{for all} \quad \phi \neq \alpha \subseteq \langle n \rangle := \{1, 2, \ldots, n\} \\ \text{and for all } D = \operatorname{diag}[d_1, d_2, \ldots, d_n] \in \mathbb{C}^{n,n}, \end{cases} \tag{1.2}$$

where $A[\alpha]$ in general denotes the principal submatrix of A determined by α, i.e., $A[\alpha] = [a_{i,j}]$ where $i, j \in \alpha$. Our main result, Theorem 6, gives two conditions on A and B, each of which is equivalent with (1.2).

On considering (1.2), we first observe from (1.1) that, on choosing $\alpha = \{i\}$ for any $i \in \langle n \rangle$, the inclusion of (1.2) implies that

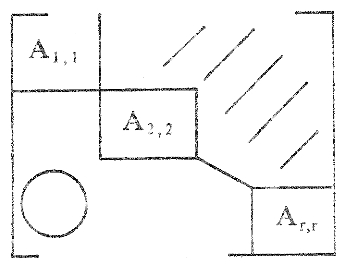
$$a_{i,i} = b_{i,i}, \quad \text{for all} \quad i \in \langle n \rangle. \tag{1.3}$$

Next, we also observe from (1.1) that the off-diagonal entries of A enter into the definition of the minimal Geršgorin set, $G(A)$, only through their moduli. This suggests that conditions which insure (1.2) will similarly depend only on the moduli of the off-diagonal entries of A and B.

2. NOTATION AND TERMINOLOGY

In this section, we collect some needed notation, terminology, and background material on minimal Geršgorin sets and ω-matrices. To begin, assumed that $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ is reducible (cf. [5, p. 46]). It is well known that there is a permutation matrix $P \in \mathbb{R}^{n,n}$ for which

$PAP^T =$



, (2.1)

where each $A_{j,j}$, $1 \leqslant j \leqslant r$, is irreducible. Here, for convenience, all 1×1 matrices are defined to be irreducible. Then, it is known (cf. [6, p. 725]) that the minimal Geršgorin set for A is precisely the union of the minimal Geršgorin sets for $A_{j,j}$, $1 \leqslant j \leqslant r$, i.e.,

$$G(A) = \bigcup_{i=1}^{r} G(A_{i,i}). \tag{2.2}$$

Next, consider any real matrix $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ satisfying

$$a_{i,i} \ \text{real}; \quad a_{i,j} \geqslant 0, \qquad i \neq j; \quad \text{for all} \quad i, j \in \langle n \rangle. \tag{2.3}$$

Such matrices are called *essentially nonnegative* (cf. [1], [5]). As a consequence of the Perron–Frobenius theory of nonnegative matrices, such a matrix satisfying (2.3) possesses a real eigenvalue v, i.e., $v \in \operatorname{spec}(A) := \{\lambda : \det(A - \lambda I) = 0\}$, which satisfies

$$v \geqslant \operatorname{Re} \lambda, \quad \text{for all} \quad \lambda \in \operatorname{spec}(A), \tag{2.4}$$

and the inclusions

$$\min_{i \in \langle n \rangle} \{ \sum_{j \in \langle n \rangle} a_{i,j} x_j / x_i \} \leqslant v \leqslant \max_{i \in \langle n \rangle} \{ \sum_{j \in \langle n \rangle} a_{i,j} x_j / x_i \}, \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n_+. \tag{2.5}$$

Furthermore, v is characterized (cf. [3, p. 201], [4]) by

$$v = \inf_{x \in \mathbb{R}^n_+} \{ \max_{i \in \langle n \rangle} [\sum_{j \in \langle n \rangle} a_{i,j} x_j / x_i] \}, \tag{2.6}$$

and, if A is irreducible,

$$v = \sum_{j \in \langle n \rangle} a_{i,j} x_j / x_i \quad \text{for all} \quad i \in \langle n \rangle, \quad \text{for some} \quad \mathbf{x} \in \mathbb{R}^n_+. \tag{2.7}$$

These facts can be used as follows. For any $A = [a_{i,j}] \in \mathbb{C}^{n,n}$ and for any $z \in \mathbb{C}$, define the real matrix $T(z) = [t_{i,j}(z)] \in \mathbb{R}^{n,n}$ by

$$t_{i,i}(z) = -|z - a_{i,i}|; \qquad t_{i,j}(z) = |a_{i,j}|, \quad i \neq j; \quad \text{for all} \quad i, j \in \langle n \rangle, \tag{2.8}$$

so that $T(z)$ satisfies (2.3). Denoting the associated eigenvalue of $T(z)$, satisfying (2.4)–(2.6), by $v(z; A)$, then $v(z; A)$ satisfies, from (2.5) and (2.6),

$$\min_{i \in \langle n \rangle} \{ -|z - a_{i,i}| + \sum_{\substack{j \in \langle n \rangle \\ j \neq i}} |a_{i,j}| x_j / x_i \} \leqslant v(z; A) \tag{2.9}$$
$$\leqslant \max_{i \in \langle n \rangle} \{ -|z - a_{i,i}| + \sum_{\substack{j \in \langle n \rangle \\ j \neq i}} |a_{i,j}| x_j / x_i \}, \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n_+,$$

and

$$v(z; A) = \inf_{x \in \mathbb{R}^n_+} \{ \max_{i \in \langle n \rangle} [-|z - a_{ii}| + \sum_{\substack{j \in \langle n \rangle \\ j \neq i}} |a_{i,j}| x_j / x_i] \}. \tag{2.10}$$

As proved in [6], the points of $G(A)$ can then be characterized in terms of the function $v(z; A)$ by means of

PROPOSITION 1 *For any $A \in \mathbb{C}^{n,n}$, $z \in G(A)$ iff $v(z; A) \geqslant 0$.*

We also include from [6] the following result.

PROPOSITION 2 *For any irreducible $A = [a_{i,j}] \in \mathbb{C}^{n,n}$, then $z \in \partial G(A)$, i.e., z is a boundary point of $G(A)$, implies that there exists an $\mathbf{x} \in \mathbb{R}^n_+$ such that*

$$|z - a_{i,i}| = \sum_{\substack{j \in \langle n \rangle \\ j \neq i}} |a_{i,j}| x_j / x_i, \quad \textit{for all} \quad i \in \langle n \rangle. \tag{2.11}$$

Next, following Engel and Schneider [2], a matrix $A \in \mathbb{C}^{n,n}$ is called an *ω-matrix* if

$$\operatorname{spec}(A[\alpha]) \cap \mathbb{R} \neq \phi, \quad \text{for all} \quad \alpha \quad \text{with} \quad \phi \neq \alpha \subseteq \langle n \rangle, \tag{2.12}$$

and, on defining

$$l(A[\alpha]) := \min\{\text{spec}(A[\alpha]) \cap \mathbb{R}\}, \tag{2.13}$$

if, for arbitrary subsets α and β of $\langle n \rangle$, one has the *nesting property*

$$\phi \neq \alpha \subseteq \beta \subseteq \langle n \rangle \quad \text{implies} \quad l(A[\beta]) \leqslant l(A[\alpha]). \tag{2.14}$$

If $\omega_{\langle n \rangle}$ denotes the set of all ω-matrices in $\mathbb{C}^{n,n}$, then, as stated in [2], $\omega_{\langle n \rangle}$ contains all the Hermitian matrices, M-matrices, as well as all totally nonnegative matrices, in $\mathbb{C}^{n,n}$.

Continuing, (cf. [2]), a reflexive and transitive relation can be defined on $\omega_{\langle n \rangle}$ as follows: if $A, B \in \omega_{\langle n \rangle}$, then $A \geqslant_\tau B$ if

$$l(A[\alpha]) \geqslant l(B[\alpha]), \quad \text{for all} \quad \alpha \quad \text{with} \quad \phi \neq \alpha \subseteq \langle n \rangle. \tag{2.15}$$

In the case that A and B are Hermitian matrices in $\mathbb{C}^{n,n}$, and thus elements of $\omega_{\langle n \rangle}$, it is easy to verify that $A \geqslant_\tau O$ iff A is positive semi-definite, and that $A \geqslant_\tau B$ if $A - B$ is positive semi-definite. In the same vein, we investigate implications of the relation $A \geqslant_\tau B$ on a subset, $\mathbb{Z}^{n,n}$, of $\omega_{\langle n \rangle}$ which contains the M-matrices of $\omega_{\langle n \rangle}$. Defining $\mathbb{Z}^{n,n}$ as the subset of $\mathbb{R}^{n,n}$ of all real matrices $A = [a_{i,j}]$ for which

$$a_{i,i} \text{ real}; \quad a_{i,j} \leqslant 0, \quad i \neq j; \quad \text{for all} \quad i, j \in \langle n \rangle, \tag{2.16}$$

then from (2.3), $A \in \mathbb{Z}^{n,n}$ iff $-A$ is essentially nonnegative. Thus, applying the characterizations of (2.4)–(2.7), it analogously follows that $A = [a_{i,j}] \in \mathbb{Z}^{n,n}$ has a real eigenvalue $l(A)$ such that

$$l(A) \leqslant \text{Re } \lambda, \quad \text{for all} \quad \lambda \in \text{spec}(A), \tag{2.17}$$

$$\min_{i \in \langle n \rangle} \{ \sum_{j \in \langle n \rangle} a_{i,j} x_j / x_i \} \leqslant l(A) \leqslant \max_{i \in \langle n \rangle} \{ \sum_{j \in \langle n \rangle} a_{i,j} x_j / x_i \}, \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n_+, \tag{2.18}$$

$$l(A) = \sup_{x \in \mathbb{R}^n_+} \{ \min_{i \in \langle n \rangle} [\sum_{j \in \langle n \rangle} a_{i,j} x_j / x_i] \}, \tag{2.19}$$

and if $A \in \mathbb{Z}^{n,n}$ is irreducible, then

$$l(A) = \sum_{j \in \langle n \rangle} a_{i,j} x_j / x_i \quad \text{for all} \quad i \in \langle n \rangle, \quad \text{for some} \quad \mathbf{x} \in \mathbb{R}^n_+. \tag{2.20}$$

It is not difficult to verify (cf. [5, p. 30]) that $l(A)$, so defined, also satisfies the nesting property of (2.14), so that $\mathbb{Z}^{n,n}$ is a subset of $\omega_{\langle n \rangle}$. This brings us to the following useful characterization of "$\geqslant_\tau$" on $\mathbb{Z}^{n,n}$.

PROPOSITION 3 *Given $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathbb{Z}^{n,n}$, then $A \geqslant_\tau B$ iff, for every $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n_+$ and for every $\phi \neq \alpha \subseteq \langle n \rangle$, there exist integers i and k in α such that*

$$\sum_{j \in \alpha} a_{k,j} y_j / y_k \geqslant \sum_{j \in \alpha} b_{i,j} x_j / x_i. \tag{2.21}$$

Proof First, assume that $A \geqslant_\tau B$. It suffices to consider only the case $\alpha = \langle n \rangle$, since the proof for any nonempty α in $\langle n \rangle$ is similar. By hypothesis,

$l(A) \geqslant l(B)$ and hence, for any $\mathbf{x}$ and any $\mathbf{y}$ in $\mathbb{R}^n_+$, we have from (2.18) that

$$\max_{i\in\langle n\rangle} \{ \sum_{j\in\langle n\rangle} a_{i,j}\, y_j/y_i \} \geqslant l(A) \geqslant l(B) \geqslant \min_{i\in\langle n\rangle} \{ \sum_{j\in\langle n\rangle} b_{i,j}\, x_j/x_i \},$$

which directly implies (2.21). Conversely, it follows from (2.21) that

$$\max_{k\in\alpha} \{ \sum_{j\in\alpha} a_{k,j}\, y_j/y_k \} \geqslant \min_{i\in\alpha} \{ \sum_{j\in\alpha} b_{i,j}\, x_j/x_i \},$$

for every $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n_+$, and every $\phi \neq \alpha \subseteq \langle n\rangle$. Hence, taking the supremum of the right side over $\mathbf{x}$ in $\mathbb{R}^n_+$ gives, from (2.19),

$$\sum_{j\in\alpha} a_{k,j}\, y_j/y_k \geqslant l(B[\alpha]) \quad \text{for some} \quad k = k(\underline{y}) \in \alpha, \quad \text{for all} \quad \mathbf{y} \in \mathbb{R}^n_+.$$

If $A[\alpha]$ is irreducible, we can, from (2.20), choose $\mathbf{y} \in \mathbb{R}^n_+$ so that

$$\sum_{j\in\alpha} a_{k,j}\, y_j/y_k = l(A[\alpha]) \quad \text{for all} \quad k \in \alpha, \quad \text{whence} \quad l(A[\alpha]) \geqslant l(B[\alpha]).$$

If $A[\alpha]$ is however reducible, there is a subset β with $\phi \neq \beta \subseteq \alpha$ for which $A[\beta]$ is irreducible, and for which $l(A[\beta]) = l(A[\alpha])$. Thus, with the above irreducible case,

$$l(A[\alpha]) = l(A[\beta]) \geqslant l(B[\beta]) \geqslant l(B[\alpha]), \tag{2.22}$$

the last inequality following from the nesting property of (2.14), and (ii) implies (i). ■

3. MAIN RESULT

We begin with

DEFINITION 4 *Given* $A = [a_{i,j}]$ *and* $B = [b_{i,j}]$ *in* $\mathbb{C}^{n,n}$, *then* $|A|$ dominates $|B|$ *if, for every* $\phi \neq \alpha \subseteq \langle n\rangle$ *for which* $B[\alpha]$ *is irreducible, and for every* $\mathbf{x}$ *and every* $\mathbf{y}$ *in* $\mathbb{R}^n_+$, *there is an* $i \in \alpha$ *for which*

$$\sum_{j\in\alpha} |a_{i,j}| y_j/y_i \geqslant \sum_{j\in\alpha} |b_{i,j}| x_j/x_i. \tag{3.1}$$

Note that (3.1) of Definition 4 is a condition like that of (2.21) of Proposition 3, but differs essentially in that (3.1) holds for the *same* i in the sums of (3.1), while (2.21) holds for possibly different i and k in the sums of (2.21).

We note from Definition 4 that $|A|$ dominates $|B|$ iff, for any non-singular diagonal matrices $S = \text{diag}[s_1, s_2, \ldots, s_n]$ and $T = \text{diag}[t_1, t_2, \ldots, t_n]$, then $|S^{-1}AS|$ dominates $|T^{-1}BT|$. As another characterization in terms of ω-matrices, we have

PROPOSITION 5 *Given* $A = [a_{i,j}]$ *and* $B = [b_{i,j}]$ *in* $\mathbb{C}^{n,n}$, *then* $|A|$ *dominates* $|B|$ *iff* $D - |B| \geqslant_\tau D - |A|$ *for all real* $D = \text{diag}[d_1, d_2, \ldots, d_n] \in \mathbb{R}^{n,n}$.

Proof Assuming first that $|A|$ dominates $|B|$, consider any real $D = \text{diag}[d_1, d_2, \ldots, d_n]$ in $\mathbb{R}^{n,n}$ and any nonempty $\alpha \subseteq \langle n\rangle$. If $B[\alpha]$ is irreducible, then, for every $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n_+$, it follows from (3.1) of Definition 4 that there is an $i \in \alpha$ for which

$$d_i - \sum_{j\in\alpha} |b_{i,j}| x_j/x_i \geqslant d_i - \sum_{j\in\alpha} |a_{i,j}|\, y_j/y_i. \tag{3.2}$$

Next, we note that $D - |A|$ and $D - |B|$ are elements of $\mathbb{Z}^{n,n}$ and, since $B[\alpha]$ is irreducible, so is $(D - |B|)[\alpha]$. Thus, from (2.20), we choose $\mathbf{x} \in \mathbb{R}^n_+$ so that

$$d_k - \sum_{j\in\alpha} |b_{k,j}|\, x_j/x_k = l((D - |B|)[\alpha]), \quad \text{for all} \quad k \in \alpha.$$

With this choice of $\mathbf{x} \in \mathbb{R}^n_+$, (3.2) implies that

$$l((D - |B|)[\alpha]) \geqslant \min_{i\in\alpha} \{d_i - \sum_{j\in\alpha} |a_{i,j}|\, y_j/y_i\}, \quad \text{for all} \quad \mathbf{y} \in \mathbb{R}_n^+.$$

Then, the characterization of (2.19) directly yields

$$l((D - |B|)[\alpha]) \geqslant l((D - |A|)[\alpha]). \tag{3.3}$$

If $B[\alpha]$ is reducible, the argument used in (2.22) of Proposition 3 can be repeated to show that (3.3) again holds, whence $D - |B| \geqslant_\tau D - |A|$. Conversely, if $D - |B| \geqslant_\tau D - |A|$ for any real $D = \operatorname{diag}[d_1, d_2, \ldots, d_n] \in \mathbb{R}^{n,n}$, assumed that α is any nonempty subset of $\langle n \rangle$ with $B[\alpha]$ irreducible, so that $(D - |B|)[\alpha]$ is irreducible. By hypotheses, $l((D - |B|)[\alpha]) \geqslant l((D - |A|)[\alpha])$ for any real $D = \operatorname{diag}[d_1, d_2, \ldots, d_n]$. For any $\mathbf{x} \in \mathbb{R}^n_+$, define the real numbers

$$\hat{d}_i := \begin{cases} \sum_{j\in\alpha} |b_{i,j}|\, x_j/x_i, & \text{for all } i \in \alpha, \\ 0 \qquad , & \text{for all } i \notin \alpha, \end{cases} \tag{3.4}$$

and set $\hat{D} := \operatorname{diag}[\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_n] \in \mathbb{R}^{n,n}$. With the definition of (3.4) and the inclusions of (2.18), it follows that

$$\hat{d}_i - \sum_{j\in\alpha} |b_{i,j}|\, x_j/x_i = 0 = l((\hat{D} - |B|)[\alpha]) \quad \text{for all } i \in \alpha. \tag{3.5}$$

On the other hand, by hypothesis and by (2.18),

$$l((\hat{D} - |B|)[\alpha]) \geqslant l((\hat{D} - |A|)[\alpha]) \geqslant \min_{i\in\alpha} \{\hat{d}_i - \sum_{j\in\alpha} |a_{i,j}|\, y_j/y_i\}, \quad \text{for any } \mathbf{y} \in \mathbb{R}^n_+. \tag{3.6}$$

Thus, combining the results of (3.5)–(3.6), there is an $i \in \alpha$ for which

$$\hat{d}_i - \sum_{j\in\alpha} |b_{i,j}|\, x_j/x_i \geqslant \hat{d}_i - \sum_{j\in\alpha} |a_{i,j}|\, y_j/y_i,$$

which implies that, for any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n_+$, there is an $i \in \alpha$ such that

$$\sum_{j\in\alpha} |a_{i,j}|\, y_j/y_i \geqslant \sum_{j\in\alpha} |b_{i,j}|\, x_j/x_i,$$

i.e., $|A|$ dominates $|B|$. ∎

Given A and B in $\mathbb{R}^{n,n}$, if there is a particular real diagonal matrix $D \in \mathbb{R}^{n,n}$ such that $D - |B| \geqslant_\tau D - |A|$, then Proposition 5 provides no information as to whether or not $|A|$ dominates $|B|$. In fact, it is possible to construct an example where $D - |B| \geqslant_\tau D - |A|$ for a particular real diagonal matrix, but where $|A|$ *fails* to dominate $|B|$. To show this, letting

$$|A| := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}; \quad |B| := \begin{bmatrix} 0 & \frac{1}{10} & \frac{1}{4} \\ 0 & 0 & 2 \\ 4 & \frac{1}{4} & 0 \end{bmatrix}; \quad |D| := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

it can be readily verified that $D - |B| \geqslant_\tau D - |A|$. On the other hand, it can be verified that $\rho(|A|) = \sqrt{2} \doteqdot 1.414\,213$ and that $\rho(|B|) \doteqdot 1.434\,467$, so that $\rho(|B|) > \rho(|A|)$. This, however, implies from Proposition 7 in the next section that $|A|$ *cannot* dominate $|B|$.

This brings us to our main result.

THEOREM 6 *Given $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathbb{C}^{n,n}$, the following are equivalent:*

i) *$G((D + B)[\alpha]) \subseteq G((D + A)[\alpha])$ for any $\phi \neq \alpha \subseteq \langle n \rangle$ and for any $D = diag[d_1, d_2, \ldots, d_n[\in \mathbb{C}^{n,n}$;*

ii) *$G((D + B)[\alpha]) \subseteq G((D + A)[\alpha])$ for any $\phi \neq \alpha \subseteq \langle n \rangle$ and for any $D = diag[d_1, d_2, \ldots, d_n] \in \mathbb{C}^{n,n}$ such that $(d_i + b_{i,i})$ is real for all $i \in \langle n \rangle$;*

iii) *$|A|$ dominates $|B|$ and $a_{i,i} = b_{i,i}$ for all $i \in \langle n \rangle$;*

iv) *$a_{i,i} = b_{i,i}$ for all $i \in \langle n \rangle$ and $D - |B| \geqslant_\tau D - |A|$ for any* **real** *$D = diag[d_1, d_2, \ldots, d_n] \in \mathbb{R}^{n,n}$.*

Proof That (i) implies (ii) is obvious. Assuming (ii), we first observe, on choosing $\alpha = \{i\}$ for any $i \in \langle n \rangle$, that (ii) implies from (1.1) that $d_i + b_{i,i} = d_i + a_{i,i}$ for any d_i such that $d_i + b_{i,i}$ is real, whence $a_{i,i} = b_{i,i}$ for all $i \in \langle n \rangle$. Next, consider any α with $\phi \neq \alpha \subseteq \langle n \rangle$ for which $B[\alpha]$ is irreducible, and any $\mathbf{x} \in \mathbb{R}^n_+$. Now, define the numbers $\tilde{d}_i$ so that

$$\tilde{d}_i + b_{i,i} = \begin{cases} \sum\limits_{\substack{j \in \alpha \\ j \neq i}} |b_{i,j}|\, x_j/x_i, & i \in \alpha, \\ 0 \qquad\qquad , & i \notin \alpha, \end{cases} \tag{3.7}$$

which implies that $\tilde{D} := \mathrm{diag}[\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n]$ satisfies the hypothesis of (ii). With this choice of the $\tilde{d}_i$'s, it follows that

$$|z - (\tilde{d}_i + b_{i,i})| = \sum_{\substack{j \in \alpha \\ j \neq i}} |b_{i,j}|\, x_j/x_i, \quad \text{for all } i \in \alpha, \tag{3.8}$$

is evidently satisfied for $z = 0$. Using the inclusions of (2.9), we see that (3.8) with $z = 0$ implies that $v(0; (\tilde{D} + B)[\alpha]) = 0$, so that, from Proposition 1, $0 \in G((\tilde{D} + B)[\alpha])$. Thus, hypothesis (ii) implies $0 \in G((\tilde{D} + A)[\alpha])$, whence from Proposition 1, $v(0; (\tilde{D} + A)[\alpha]) \geqslant 0$. With the second inequality of (2.9), there is, for any $\mathbf{y} \in \mathbb{R}^n_+$, an $i \in \alpha$ for which

$$0 \leqslant v(0; (\tilde{D} + A)[\alpha]) \leqslant -|\tilde{d}_i + a_{i,i}| + \sum_{\substack{j \in \tau \\ j \neq i}} |a_{i,j}|\, y_j/y_i.$$

But as $a_{i,i} = b_{i,i}$ for all $i \in \langle n \rangle$, the above inequality becomes, with the definition of (3.7), just

$$\sum_{j\in\alpha} |b_{i,j}|\, x_j/x_i \leqslant \sum_{j\in\alpha} |a_{i,j}|\, y_j/y_i,$$

i.e., $|A|$ dominates $|B|$, and (ii) implies (iii). That (iii) implies (iv) follows directly from Proposition 5.

Finally, we show that (iv) implies (i). Assuming (iv), consider any α with $\phi \neq \alpha \subseteq \langle n \rangle$ and any $D = \mathrm{diag}[d_1, d_2, \ldots, d_n] \in \mathbb{C}^{n,n}$. To establish (i), it suffices to show that if $z \in \partial G((D + B)[\alpha])$, then $z \in G((D + A)[\alpha])$. Moreover, because of (2.2), we may assume that $(D + B)[\alpha]$ is irreducible. Thus, as $z \in \partial G((D + B)[\alpha])$, it follows from Proposition 2 that there is an $\mathbf{x} \in \mathbb{R}^n_+$ for which (cf. (2.11))

$$|z - (d_i + b_{i,i})| = \sum_{\substack{j\in\langle n\rangle\\ j\neq i}} |b_{i,j}|\, x_j/x_i, \quad \text{for all } i \in \langle n \rangle. \tag{3.9}$$

Next, with the definition of the real numbers $\hat{d}_i$ in (3.4), we see that the associated matrix $\hat{D} := \mathrm{diag}[\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_n]$ is real, and the hypothesis of (iv) gives us that $\hat{D} - |B| \geqslant_\tau \hat{D} - |A|$. Now, from Proposition 3, this implies that, for any $\mathbf{y} \in \mathbb{R}^n_+$, there exist integers i and k in α such that

$$\hat{d}_i - \sum_{j\in\alpha} |b_{i,j}|\, x_j/x_i \geqslant \hat{d}_k - \sum_{j\in\alpha} |a_{k,j}|\, y_j/y_k.$$

Hence, from the definition of the $\hat{d}_j$'s in (3.4) and the hypothesis that $a_{i,i} = b_{i,i}$ for all $i \in \langle n \rangle$, the above inequality reduces to

$$\sum_{\substack{j\in\alpha\\ j\neq k}} |a_{k,j}|\, y_j/y_k \geqslant \sum_{\substack{j\in\alpha\\ j\neq k}} |b_{k,j}|\, x_j/x_k.$$

Coupling this with (3.9) and again using the fact that $a_{k,k} = b_{k,k}$ then yield

$$\max_{\mathbf{y}\in\mathbb{R}^n_+} \Big\{ \sum_{\substack{j\in\alpha\\ j\neq k}} |a_{k,j}|\, y_j/y_k - |z - (d_k + a_{k,k})| \Big\} \geqslant 0.$$

Thus, from (2.10), we deduce that $v(z; (D + A)[\alpha]) \geqslant 0$, whence, from Proposition 1, $z \in G((D + A)[\alpha])$, i.e., (iv) implies (i). ∎

4. PROPERTIES ASSOCIATED WITH THE RELATION 'A' DOMINATES 'B'

We begin with any easy consequence of Definition 4. For notation, let $\rho(C)$ denote the spectral radius of any $C \in \mathbb{C}^{n,n}$, i.e., $\rho(C) := \max\{|\lambda| : \lambda \in \mathrm{spec}\,(C)\}$.

PROPOSITION 7 *Given $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathbb{C}^{n,n}$, then $|A|$ dominates $|B|$ implies that*

$$\rho((|A|)[\alpha]) \geqslant \rho((|B|)[\alpha]) \quad \textit{for any} \quad \phi \neq \alpha \subseteq \langle n \rangle. \tag{4.1}$$

Proof If $|A|$ dominates $|B|$, then, from Proposition 5 with $D = 0$, we have that $-|B| \geqslant_\tau -|A|$, or equivalently, that

$$l(-(|B|)[\alpha]) \geqslant l(-(|A|)[\alpha]) \quad \text{for any} \quad \phi \neq \alpha \subseteq \langle n \rangle. \tag{4.2}$$

But, from the Perron–Frobenius theory of nonnegative matrices, it is easy to see from (2.19) that $l(-(|B|)[\alpha]) = -\rho((|B|)[\alpha])$, and similarly $l(-(|A|)[\alpha]) = \rho((|A|)[\alpha])$, whence (4.2) implies (4.1). ■

For the next result, we write in the usual notation of nonnegative matrices that $|A| \geqslant |B|$ if $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathbb{C}^{n,n}$ satisfy

$$|a_{i,j}| \geqslant |b_{i,j}| \quad \text{for all} \quad i, j \in \langle n \rangle. \tag{4.3}$$

PROPOSITION 8 *Given $A = [a_{i,j}]$ and $B = [b_{i,j}]$ in $\mathbb{C}^{n,n}$ satisfying* (1.3), *then $|A| \geqslant |B|$ implies that $|A|$ dominates $|B|$.*

Proof If $|A| \geqslant |B|$, then for any nonempty $\alpha \subseteq \langle n \rangle$, for any real $D = \mathrm{diag}[d_1, d_2, \ldots, d_n] \in \mathbb{R}^{n,n}$, and for any $\mathbf{x} \in \mathbb{R}^n_+$,

$$d_i - \sum_{j \in \alpha} |b_{i,j}|\, x_j/x_i \geqslant d_i - \sum_{j \in \alpha} |a_{i,j}|\, x_j/x_i, \quad \text{for all} \quad i \in \alpha. \tag{4.4}$$

Hence, since $D - |A|$ and $D - |B|$ are elements of $\mathbb{Z}^{n,n}$, it follows from (4.4) and the characterization of (2.19) that

$$l((D - |B|)[\alpha]) \geqslant l((D - |A|)[\alpha]) \quad \text{for all nonempty} \quad \alpha \in \langle n \rangle,$$

whence $D - |B| \geqslant_\tau D - |A|$ for all real $D = \mathrm{diag}[d_1, d_2, \ldots, d_n] \in \mathbb{R}^{n,n}$. Thus, applying Propositions 5, $|A|$ dominates $|B|$. ■

Finally, we remark that the converse of Propositions 7 is false. As a counterexample, consider

$$|A| := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad |B| := \begin{bmatrix} 0 & 0.08 & \frac{1}{4} \\ 0 & 0 & 2 \\ 4 & \frac{1}{4} & 0 \end{bmatrix}.$$

By direct computation, it can be verified that (4.1) is valid in this case for all $\phi \neq \alpha \subseteq \langle 3 \rangle$. However, $|A|$ does *not* dominate $|B|$. To see this, choose $\hat{\alpha} = \langle 3 \rangle$, and $\hat{\mathbf{x}} = [0.30, 1, 1]^T$ and $\hat{\mathbf{y}} = [0.9091, 0.5001, 1]^T$ in $\mathbb{R}^3_+$. Then, as can be verified, the inequality of (3.1) of Definition 4 fails for every $i \in \langle 3 \rangle$. In the same vein, the converse of Proposition 8 is easily seen to be false, since, with

$$|A| = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad |B| = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$|A|$ dominates $|B|$ and $|B|$ dominates $|A|$, but $|A| \neq |B|$.

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