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On Incomplete Polynomials

by

E. B. Saff* and R. S. Varga**

In [2], G. G. Lorentz recently raised interesting new questions concerning the uniform approximation of continuous functions on [0,1] by sequences of polynomials of the special form $\sum_{k=[\theta n]}^{n} a_k x^k$, θ fixed with $0 < \theta < 1$, n arbitrary, $n \geq 0$. Some convergence properties of sequences of such incomplete polynomials were studied by the authors [6], and by Kemperman and Lorentz [1]. In this present paper, we investigate the analog of the classical Chebyshev polynomials for this new approximation problem.

For notation, for each nonnegative integer n, let π_n denote as usual the set of all complex polynomials of degree at most n. Then, for any nonnegative integers s and m, the set $\pi_{s,m}$ of polynomials is defined by

(1) $\pi_{s,m} := \{x^s q_m(x) : q_m \in \pi_m\}$,

and, clearly, $\pi_{s,m} \subseteq \pi_{s+m}$. A polynomial p in $\pi_{s,m}$ is called an <u>incomplete polynomial of type</u> (s,m). Next, for any continuous function g on a compact set K in the complex plane, we further set

$$\|g\|_K := \max\{|g(t)| : t \in K\}.$$

Concerning the location of points where $p \in \pi_{s,m}$ attains its maximum absolute value in [0, +1], we have proved

<u>Proposition 1</u>. (Lorentz [2], and Saff and Varga [6]). Let $0 \not\equiv p \in \pi_{s,m}$, where $s+m > 0$.

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If $|p(\xi)| = \|p\|_{[0,1]}$ with $\xi \in [0,1]$, then

(2) $(s/(s+m))^2 \leq \xi \leq 1.$

Furthermore, the first inequality of (2) is best possible in the following sense: For each fixed θ with $0 \leq \theta \leq 1$ and for every infinite sequence of ordered pairs $\{(s_i, m_i)\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} (s_i + m_i) = \infty$ and $\lim_{i\to\infty} \frac{s_i}{s_i + m_i} = \theta$, there exist polynomials $\{p_i\}_{i=1}^{\infty}$ with $0 \not\equiv p_i \in \pi_{s_i,m_i}$, and points $\{\xi_i\}_{i=1}^{\infty}$ in $[0,1]$ with $|p_i(\xi_i)| = \|p_i\|_{[0,1]}$, such that

(3) $\lim_{i\to\infty} \xi_i = \theta^2.$

We remark that the first inequality of (2) is a simple consequence of a result of G. G Lorentz [2], and that the sharpness of this inequality is proved in [6], using a suitably modified sequence of Jacobi polynomials.

We now study an analogue of the classical Chebyshev polynomials for the set $\pi_{s,m}$.

Proposition 2. For each pair of nonnegative integers (s,m), there exists a unique monic polynomial $Q_{s,m}$ in $\pi_{s,m}$ of exact degree s + m such that

(4) $\|Q_{s,m}\|_{[0,1]} = \inf\{\|x^{s+m} - g\|_{[0,1]}: g \in \pi_{s,m-1}\} := E_{s,m},$

(where $\pi_{s,m-1}$ denotes the set $\{0\}$ if m = 0). Furthermore, for s + m > 0, $Q_{s,m}$ has an alternation set in [0,1] of precisely m + 1 distinct points $\xi_j^{(s,m)}$, $0 \leq j \leq m$, with

(5) $(s/(s+m))^2 \leq \xi_0^{(s,m)} < \xi_1^{(s,m)} < \ldots < \xi_m^{(s,m)} = 1,$

for which

(6) $Q_{s,m}(\xi_j^{(s,m)}) = (-1)^{m-j} E_{s,m}$, $0 \leq j \leq m.$

Moreover, for $s + m > 0$, there are no points in $[0,1]$, other than the $\xi_j^{(s,m)}$, for which $|Q_{s,m}(x)| = \|Q_{s,m}\|_{[0,1]} = E_{s,m}$.

Proof. As Proposition 2 is clearly true if $s = 0$ or if $m = 0$, assume $s > 0$ and $m > 0$, and let $p^* \in \pi_{s,m}$ be any monic polynomial of degree $s + m$ for which

(7) $\|p^*\|_{[0,1]} = E_{s,m}$.

Expressing $p^*(x)$ as $p^*(x) = x^{s+m} - g^*(x)$, $g^* \in \pi_{s,m-1}$, then Proposition 1 shows that p^* also solves the extremal problem

(8) $\|p^*\|_{[\lambda,1]} = \inf\{\|x^{s+m} - g\|_{[\lambda,1]}: g \in \pi_{s,m-1}\}$,

where $\lambda := (s/(s+m))^2 > 0$. As $\pi_{s,m-1}$ is a linear space of dimension m which satisfies the Haar condition on the interval $[\lambda,1]$, it is well-known (cf. Meinardus [2, p. 20]) that g^* (and hence p^*) is unique, and has an associated alternation set for the difference $x^{s+m} - g^* = p^*$, consisting of at least $m + 1$ distinct points in $[\lambda,1]$. If, however, $|p^*(x)|$ attains its maximum in more than $m + 1$ distinct points of $[0,1]$, then the derivative of p^*, which belongs to $\pi_{s-1,m}$, would vanish in at least $s + m$ points of $[0,1]$ and would, consequently, be identically zero, contradicting the fact that p^* is monic in π_{s+m}. Thus, there are precisely $m + 1$ distinct points in $[0,1]$ where $|p^*(x)|$ attains its maximum. A similar argument can be used to prove that $\xi_m^{(s,m)} = 1$.

From the unique monic polynomials $Q_{s,m}$ of Proposition 2, we then define

(9) $T_{s,m}(x) := Q_{s,m}(x)/\|Q_{s,m}\|_{[0,1]} = Q_{s,m}(x)/E_{s,m}$.

to be the (normalized) constrained Chebyshev polynomial of degree $s + m$

associated with the set $\pi_{s,m}$ on $[0,1]$. Now, in the following special case $m = 0$, the quantities $Q_{s,m}(x)$, $E_{s,m}$, $\xi_j^{(s,m)}$, and $T_{s,m}(x)$ can easily be determined:

$$(10)\quad Q_{s,0}(x) = T_{s,0}(x) = x^s;\ E_{s,0} = 1;\ \xi_0^{(s,0)} = 1.$$

Next, if $T_n(t) = \cos[n \cos^{-1} t]$ for $|t| \le 1$ denotes the classical Chebyshev polynomial (of the first kind) of degree n, then for $s = 0$, we similarly easily deduce that

$$(11)\quad \begin{cases} Q_{0,m}(x) = T_m(2x - 1)/2^{2m-1} \quad ; \quad T_{0,m}(x) = T_m(2x - 1); \\ E_{0,m} = 2^{1-2m} \quad ; \quad \xi_j^{(0,m)} = [1-\cos(j\pi/m)]/2 \quad , \quad 0 \le j \le m. \end{cases}$$

Thus, the constrained Chebyshev polynomial $T_{0,m}(x)$ for $\pi_{0,m}$ on $[0,1]$ reduces to the (translated) classical Chebyshev polynomial $T_m(2x-1)$.

For the special cases $s = 1$ and $m = 1$, we have

Proposition 3. For any $m \ge 1$, set $x_{1,m} := -\cos(\pi/2m)$. Then,

$$(12)\quad \begin{cases} Q_{1,m-1}(x) = T_m[(1 - x_{1,m})x + x_{1,m}]/2^{m-1}(1-x_{1,m})^m;\ E_{1,m-1} = 2^{1-m}(1-x_{1,m})^{-m}; \\ T_{1,m-1}(x) = T_m[(1-x_{1,m})x+x_{1,m}];\ \xi_j^{(1,m-1)} = \dfrac{\cos[(\frac{m-1-j}{m})\pi] - x_{1,m}}{1 - x_{1,m}},\ 0\le j\le m-1. \end{cases}$$

Similarly, for any $s > 1$, let α_s be the unique positive root of

$$(13)\quad \frac{\alpha^s(s-1)^{s-1}}{s^s} + \alpha - 1 = 0.$$

Then,

$$(14)\quad \begin{cases} Q_{s-1,1}(x) = x^{s-1}(x - \alpha_s) \quad ; \quad E_{s-1,1} = 1 - \alpha_s; \\ T_{s-1,1}(x) = x^{s-1}(x - \alpha_s)/(1 - \alpha_s);\ \xi_0^{(s-1,1)} = \dfrac{(s-1)\alpha_s}{s};\ \xi_1^{(s-1,1)} = 1. \end{cases}$$

Proof. By definition, $x_{1,m}$ is the least (simple) zero of T_m, so that $T_m[(1-x_{1,m})x+x_{1,m}] \in \pi_{1,m-1}$. The rest of (12) follows from well-known properties of Chebyshev polynomials (cf. Rivlin [5]). Next, for $s > 1$, (13) has a unique positive root, α_s, by Descartes' Rules of Signs, and, moreover, $\alpha_s < 1$. To determine $Q_{s-1,1}(x)$, we note from Proposition 2 that its alternation set consists of two points, $0 < \xi_0^{(s-1,1)} < \xi_1^{(s-1,1)} = 1$. Hence, on writing $Q_{s-1,1}(x) = x^{s-1}(x-\alpha)$, we find that $Q'_{s-1,1}(x) = 0$ only if $x = 0$ or if $x = \bar{x} = \alpha(s-1)/s$. Setting $Q_{s-1,1}(\bar{x}) = -Q_{s-1,1}(1)$ gives exactly (13), and as $|Q_{s-1,1}(1)| = E_{s-1,1}$, the rest of (14) follows.

We remark that, using (13) and (14), it can be shown that

$$(15) \quad \xi_0^{(s-1,1)} = \frac{1}{1 + \frac{(K+1)}{s} + \mathcal{O}(\frac{1}{s^2})}, \text{ as } s \to \infty$$

where K is the unique positive root of the equation $Ke^{K+1} = 1$ (so that $K \doteq 0.278\ 465$).

Next, we show that, outside the interval $(\xi_0^{(s,m)}, 1)$, the polynomials $T_{s,m}(x)$ dominate the growth of polynomials in $\pi_{s,m}$.

Proposition 4. If $p \in \pi_{s,m}$, and if $M \geq \max\{|p(\xi_k^{(s,m)})| : 0 \leq k \leq m\}$, then

$$(16) \quad |p(x)| \leq M|T_{s,m}(x)|$$

for all real x outside the interval $(\xi_0^{(s,m)}, 1)$. Furthermore, for any positive integer ν,

$$(17) \quad |p^{(\nu)}(x)| \leq M|T_{s,m}^{(\nu)}(x)| \text{ for all } x \notin (0,1).$$

Proof. First, set $h(x) := x^s \prod_{j=0}^{m} (x - \xi_j^{(s,m)})$. Then, by means of the Lagrange interpolation formula (applied in the points $\xi_j^{(s,m)}$) and the alternation property of $Q_{s,m}$ in Proposition 2, it follows from (6) and (9) that

$$(18)\quad T_{s,m}(x) = \sum_{k=0}^{m} \frac{T_{s,m}(\xi_k)\, h(x)}{h'(\xi_k)(x-\xi_k)} = \sum_{k=0}^{m} \frac{(-1)^{m-k} h(x)}{h'(\xi_k)(x-\xi_k)},$$

where, for convenience, we have omitted the superscript (s,m) on the points ξ_k. Now, for $p \in \pi_{s,m}$, we also have

$$(19)\quad p(x) = \sum_{k=0}^{m} \frac{p(\xi_k)\, h(x)}{h'(\xi_k)(x-\xi_k)},$$

so that, by hypothesis,

$$(20)\quad |p(x)| \le \sum_{k=0}^{m} \frac{|p(\xi_k)|\; |h(x)|}{|h'(\xi_k)|\cdot|x-\xi_k|} \le M \sum_{k=0}^{m} \frac{|h(x)|}{|h'(\xi_k)|\cdot|x-\xi_k|}.$$

For $0 \le x \le \xi_0$, then $|h'(\xi_k)| = |\xi_k^s \prod_{\substack{j=0\\ j\ne k}}^{m} (\xi_k - \xi_j)| = (-1)^{m-k}\, h'(\xi_k)$, and $|h(x)| = (-1)^{m+1} h(x)$, so that, from (20),

$$(21)\quad |p(x)| \le M \sum_{k=0}^{m} \frac{(-1)^{m+1}\; h(x)}{(-1)^{m-k}\; h'(\xi_k)(\xi_k - x)} \qquad \text{for } 0 \le x \le \xi_0.$$

But from (18), the sum on the right side of (21) equals $|T_{s,m}(x)|$, whence

$$(22)\quad |p(x)| \le M|T_{s,m}(x)| \qquad \text{for all } 0 \le x \le \xi_0.$$

Similarly, (22) holds for $x \le 0$, as well as for $x \ge \xi_m = 1$, which establishes (16). Finally, to establish (17), simply differentiate the equation in (19) ν times, and argue as in the first part of the proof.

As an immediate consequence of Proposition 4, we can give a sharpened version for the first inequality of (2) of Proposition 1.

Corollary 5. For any $s \geq 1$, let $\mu_{s,m}$ be the unique negative value of x for which $|T_{s,m}(x)| = 1$. If $0 \not\equiv p \in \pi_{s,m}$ and if $|p(\xi)| = \|p\|_{[0,1]}$ with ξ real, then either

(23) $$\xi \leq \mu_{s,m} \text{ or } \xi \geq \xi_0^{(s,m)}.$$

Note that the second inequality of (23) of Corollary 5 asserts that, among all $0 \not\equiv p \in \pi_{s,m}$ with $s + m > 0$, the constrained Chebyshev polynomial $T_{s,m}(x)$ attains its maximum absolute value on $[0,1]$ at the least point in $[0,1]$. This observation, when coupled with the second part of Proposition 1, gives

Corollary 6. For each fixed θ with $0 \leq \theta \leq 1$, and for every infinite sequence of ordered pairs $\{(s_i, m_i)\}_{i=1}^{\infty}$ with $\lim_{i\to\infty}(s_i+m_i) = \infty$ and $\lim_{i\to\infty} \frac{s_i}{s_i+m_i} = \theta$, then

(24) $$\lim_{i\to\infty} \xi_0^{(s_i,m_i)} = \theta^2.$$

We remark that, with the definition of $r(\theta)$ in [6], the same hypotheses of Corollary 6 similarly yield (cf. (23))

(25) $$\lim_{i\to\infty} \mu_{s_i,m_i} = (1 + r(\theta))/2.$$

Further properties of the $\xi_0^{(s,m)}$ are given in

Proposition 7. For all $s \geq 0$ and $m > 0$,

(26) $$\xi_0^{(s,m)} < \xi_0^{(s+1,m-1)},$$

and for all $s > 0$ and $m > 0$,

(27) $$\xi_0^{(ks,km)} < \xi_0^{(s,m)} \quad \text{for all } k > 1.$$

Thus, for all $s > 0$ and $m > 0$, the first inequality of (5) can be sharpened to

(28) $$(s/(s+m))^2 < \xi_0^{(s,m)}.$$

Proof. By definition, $\pi_{s+1,m-1} \subseteq \pi_{s,m}$, so that $\xi_0^{(s+1,m-1)} \geq \xi_0^{(s,m)}$ by the remark above following Corollary 5. Now, the monic polynomial $Q_{s+1,m-1}(x)$ has, from Proposition 2, precisely m alternation points in $[0,1]$, while $Q_{s,m}$ has, on the other hand, m + 1 alternation points in $[0,1]$. Using a slight variation of the perturbation argument given in Rivlin [5, p. 26], it can however be shown that $\{Q_{s+1,m-1}(x)+\lambda x^s t_m(x)\} \in \pi_{s,m}$, for suitable $t_m \in \pi_m$ and small λ, attains its maximum absolute value in $[0,1]$ in a point ξ with $\xi < \xi_0^{(s+1,m-1)}$, whence $\xi_0^{(s,m)} < \xi_0^{(s+1,m-1)}$. Similarly, if $p \in \pi_{s,m}$, then p^k is an element of $\pi_{ks,km}$ for any $k > 1$, and, moreover, p and p^k attain their maximum absolute values in $[0,1]$ in the same points, say ξ_ℓ. The above perturbation argument again shows that $\xi_0^{(ks,km)} < \xi_\ell$ for any ℓ, whence, on choosing $p = Q_{s,m}$, then $\xi_0^{(ks,km)} < \xi_0^{(k,m)}$.

To indicate the actual values of $\xi_0^{(s,m)}$ for some small values of s and m, we give below in Table 1 the values of $\xi_0^{(s,m)}$ (rounded to three decimal places) for all $s + m \leq 8$.

m+s \ s	0	1	2	3	4	5	6	7	8
0	0								
1	0	1							
2	0	.414	1						
3	0	.196	.596	1					
4	0	.113	.355	.693	1				
5	0	.073	.232	.465	.752	1			
6	0	.051	.163	.330	.544	.792	1		
7	0	.037	.121	.245	.408	.603	.821	1	
8	0	.029	.093	.189	.316	.470	.649	.843	1

Table 1: $\xi_0^{(s,m)}$ for $s + m \leq 8$.

Concerning the limiting behavior of $E_{s,m} = \|Q_{s,m}\|_{[0,1]}$ of (4), we now prove

Proposition 8. For each fixed θ with $0 \leq \theta \leq 1$, and for every infinite sequence of ordered pairs $\{(s_i, m_i)\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} (s_i+m_i) = \infty$ and $\lim_{i\to\infty} \frac{s_i}{s_i+m_i} = \theta$, then

$$(29) \quad \lim_{i\to\infty} (E_{s_i,m_i})^{\frac{1}{s_i+m_i}} = \frac{(1+\theta)^{1+\theta}(1-\theta)^{1-\theta}}{4} .$$

Proof. We first compare the monic polynomial $Q_{s,m}$ in $\pi_{s,m}$ of Proposition 2 with the following modified Jacobi polynomial

$$U_{s,m}(x) := \binom{2m+2s}{m}^{-1} x^s P_m^{(0,2s)}(2x-1)$$

which is also in $\pi_{s,m}$. It follows from Szegö [7, p. 63, eq. (4.21.6)] that $U_{s,m}$, so defined, is also monic of exact degree $s + m$. Furthermore, it again follows from Szegö [7, p. 163, Thm. 7.2] that $U_{s,m}(x)$ attains its maximum absolute value on $[0,1]$ at $x = 1$, i.e.,

$$(30) \quad \|U_{s,m}\|_{[0,1]} = |U_{s,m}(1)| = \frac{|P_m^{(0,2s)}(1)|}{\binom{2m+2s}{m}} = \binom{2m+2s}{m}^{-1} ,$$

since $|P_m^{(0,2s)}(1)| = 1$ (cf. [7, p. 58]). Since $E_{s,m} \leq \|U_{s,m}\|_{[0,1]}$ from (4) of Proposition 2, it follows from the hypotheses and Stirling's formula that

$$(31) \quad \limsup_{i\to\infty} (E_{s_i,m_i})^{\frac{1}{s_i+m_i}} \leq \lim_{i\to\infty} \binom{2m_i+2s_i}{m_i}^{-\frac{1}{s_i+m_i}} = \frac{(1+\theta)^{1+\theta}(1-\theta)^{1-\theta}}{4} .$$

On the other hand,

$$(32) \quad \int_0^1 U_{s,m}^2(x)dx \leq \int_0^1 Q_{s,m}^2(x)dx \leq E_{s,m}^2$$

because of the orthogonality property of $P_m^{(0,2s)}(2x-1)$ with respect to the weight function x^{2s} on $[0,1]$ (cf. Szegö [7, p. 39]). But, from known properties of Jacobi polynomials (cf. [7, p. 68]),

$$(33)\quad \int_0^1 U_{s,m}^2(x)\,dx = (2m+2s+1)^{-1}\binom{2m+2s}{m}^{-2},$$

so that from (32), we have similarly deduced

$$(34)\quad \frac{(1+\theta)^{1+\theta}(1-\theta)^{1-\theta}}{4} \le \liminf_{i\to\infty} \left(E_{s_i,m_i}\right)^{\frac{1}{s_i+m_i}}.$$

Thus, (31) and (34) together imply (29).

The result of Proposition 8 is used in proving

Proposition 9. For each fixed θ with $0 \le \theta < 1$, let

$$(35)\quad z = \psi(w) = \frac{1+\theta^2}{2} + \frac{(1-\theta^2)}{2}\left\{\frac{w+w^{-1}}{2}\right\},$$

map the exterior of the circle $|w| = 1$ in the w-plane onto the exterior of the interval $[\theta^2,1]$ in the z-plane, and let $w = \phi(z)$ denote the inverse of ψ. For every infinite sequence of ordered pairs $\{(s_i,m_i)\}_{i=1}^{\infty}$ with

(36) $$\lim_{i\to\infty}(s_i+m_i) = \infty \text{ and } \lim_{i\to\infty}\frac{s_i}{s_i+m_i} = \theta,$$

then

(37) $$\lim_{i\to\infty}|Q_{s_i,m_i}(z)|^{\frac{1}{s_i+m_i}} = \frac{(1+\theta)^{1+\theta}(1-\theta)^{1-\theta}}{4}\ |\phi(z)|\cdot\left|\frac{(1-\theta)\phi(z)+(1+\theta)}{(1+\theta)\phi(z)+(1-\theta)}\right|^{\theta}$$

uniformly on any compact set exterior to $[\theta^2,1]$.

<u>Proof</u>. Setting $\theta_i := s_i/(s_i+m_i)$, define as above the maps $\psi_i(w)$ and $\phi_i(z)$, where θ is replaced by θ_i. Furthermore, set

$$K := \mathbb{C}^*\setminus[\theta^2,1] \quad\text{and}\quad K_i := \mathbb{C}^*\setminus[\theta_i^2,1],$$

where $\mathbb{C}^*$ denotes the extended complex plane, and set

$$\Delta := \frac{(1+\theta)^{1+\theta}(1-\theta)^{1-\theta}}{4} \quad ; \quad \Delta_i := \frac{(1+\theta_i)^{1+\theta_i}(1-\theta_i)^{1-\theta_i}}{4}.$$

Recalling the (unnormalized) monic polynomials $Q_{s,m}$ of Proposition 2, we now study the functions

(38) $$u_i(z) := \frac{1}{(s_i+m_i)}\ln|Q_{s_i,m_i}(z)|$$

and

(39) $$v_i(z) := \ln\Delta_i + \ln|\phi_i(z)| + \theta_i \ln\left|\frac{(1-\theta_i)\phi_i(z)+(1+\theta_i)}{(1+\theta_i)\phi_i(z)+(1-\theta_i)}\right|.$$

Since all the nontrivial zeros of $Q_{s_i,m_i}(z)$ lie in $[\theta_i^2,1]$, each $u_i(z)$ is harmonic in K_i, except at $z = 0$ and at $z = \infty$. Furthermore, near $z = 0$, we have

(40) $$u_i(z) = \theta_i \ln|z| + h_i(z),$$

where $h_i(z)$ is harmonic at $z = 0$, while in a neighborhood of infinity, we have,

$$(41) \quad u_i(z) = \ln|z| + g_i(z),$$

where $g_i(z)$ is harmonic at ∞ and $g_i(\infty) = 0$.

We also note that each $v_i(z)$ is harmonic in K_i except when $z = \infty$ and when $\phi_i(z) = -(1+\theta_i)/(1-\theta_i)$, i.e., when $z = 0$. Furthermore, near $z = 0$ we have

$$(42) \quad v_i(z) = \theta_i \ln|z| + \hat{h}_i(z),$$

where $\hat{h}_i(z)$ is harmonic at $z = 0$, and in a neighborhood of infinity we have

$$(43) \quad v_i(z) = \ln|z| + \hat{g}_i(z),$$

where $\hat{g}_i(z)$ is harmonic at ∞ and $\hat{g}_i(\infty) = 0$.

From the above, we see that the functions

$$(44) \quad d_i(z) := u_i(z) - v_i(z)$$

are each harmonic in K_i, even at $z = 0$ and $z = \infty$. We also observe from (39) that as z approaches any point on the segment $[\theta_i^2, 1]$, the function $v_i(z)$ approaches $\ln\Delta_i$, and hence

$$(45) \quad \limsup_{\substack{z \to [\theta_i^2, 1] \\ z \in K_i}} d_i(z) \le \frac{1}{s_i+m_i} \ln E_{s_i,m_i} - \ln \Delta_i .$$

By assumption (36) and Proposition 8, we have

(46) $$\lim_{i\to\infty} \left\{\frac{1}{(s_i+m_i)} \ln E_{s_i,m_i} - \ln \Delta_i\right\} = 0,$$

and hence, by the maximum principle, the harmonic functions $d_i(z)$ are, for i sufficiently large, uniformly bounded from above on any closed set in K; indeed, since $\lim_{i\to\infty} \theta_i=\theta$, such a set must lie in K_i for each i sufficiently large. Hence, the $d_i(z)$ form a normal family of harmonic functions in K.

Letting d(z) denote any limit function of the $d_i(z)$, it follows from (45) and (46) that $d(z) \le 0$ in K. But as $d_i(\infty) = g_i(\infty) - \hat{g}_i(\infty) = 0$, we have $d(z) \equiv 0$ in K. Thus, $d_i(z) \to 0$ uniformly on any closed set in K. Now, since by condition (36),

(47) $$\lim_{i\to\infty} v_i(z) = \ln \Delta + \ln|\varphi(z)| + \theta \ln \left|\frac{(1-\theta)\varphi(z) + (1+\theta)}{(1+\theta)\varphi(z) + (1-\theta)}\right| := v(z),$$

uniformly on any closed set in $K\setminus\{0,\infty\}$, we have that $u_i(z) \to v(z)$ uniformly on closed sets in $K\setminus\{0,\infty\}$. Consequently,

$$|Q_{s_i,m_i}(z)|^{1/(s_i+m_i)} = e^{u_i(z)} \to e^{v(z)},$$

uniformly on any compact set in K, which, with (47), gives the desired equation (37).

We remark that by using steepest descent methods, the same limit function (37) was obtained for certain sequences of modified Jacobi polynomials [6]. The above proof of Proposition 9 can be viewed as furnishing an alternate proof of this fact.

Our next results concern the m nontrivial (i.e., nonzero) zeros, $\sigma_i^{(s,m)}$, of the constrained Chebyshev polynomial $T_{s,m}(x)$. From the alternation characterization of $Q_{m,s}(x)$ in Proposition 2, it follows that, for $m > 0$, these zeros satisfy (cf. (5))

(48) $$(s/(s+m))^2 \le \xi_0^{(s,m)} < \sigma_1^{(s,m)} < \xi_1^{(s,m)} < \sigma_2^{(s,m)} < \cdots < \sigma_m^{(s,m)} < \xi_m^{(s,m)} = 1.$$

Lemma 10. For $0 \not\equiv p(x) = x^s g_m(x)$ in $\pi_{s,m}$, $m > 0$, assume that there exist m points $\tilde{\xi}_i$, with $0 < \tilde{\xi}_0 < \tilde{\xi}_1 < \cdots \tilde{\xi}_{m-1} < 1 =: \tilde{\xi}_m$ for which $p'(\tilde{\xi}_j) = 0$ for $0 \le j < m$, and for which

$$\text{(49)} \quad \operatorname{sgn} p(\tilde{\xi}_j) + \operatorname{sgn} p(\tilde{\xi}_{j+1}) = 0 \quad , \quad 0 \le j < m,$$

and

$$\text{(50)} \quad |p(\tilde{\xi}_0)| = \|p\|_{[0,1]} \ge |p(\tilde{\xi}_1)| \ge \cdots \ge |p(\tilde{\xi}_m)|.$$

Let $\tilde{\sigma}_j$ denote the m (simple) zeros of g_m, so that

$$\text{(51)} \quad \tilde{\xi}_0 < \tilde{\sigma}_1 < \tilde{\xi}_1 < \tilde{\sigma}_2 < \cdots < \tilde{\sigma}_m < \tilde{\xi}_m = 1.$$

If $\|p\|_{[0,1]} > |p(\tilde{\xi}_m)|$, then

$$\text{(52)} \quad \sigma_i^{(s,m)} < \tilde{\sigma}_i \text{ for all } 1 \le i \le m.$$

Proof. Without loss of generality, we may assume that $p(x) = x^s \prod_{i=1}^{m} (x-\tilde{\sigma}_i)$. Next, for $\epsilon > 0$ and any i with $1 \le i \le m$, set

$$\text{(53)} \quad p_{i,\epsilon}(x) := \left(x^s \prod_{\substack{j=1 \\ j \ne i}}^{m} (x-\tilde{\sigma}_j) \right) (x-\tilde{\sigma}_i+\epsilon),$$

where the product in (53) is defined to be unity if $m = 1$. By definition, $p_{i,\epsilon}$ is in $\pi_{s,m}$, and is monic of exact degree $m + s$. For ϵ sufficiently small, a perturbation argument shows that the points $\hat{\xi}_j$ where $p'_{i,\epsilon}(\hat{\xi}_j) = 0$, $0 \le j < m$, satisfy

$$\text{(54)} \quad \hat{\xi}_j = \tilde{\xi}_j - \frac{\epsilon |p(\tilde{\xi}_j)|}{(\tilde{\xi}_j - \sigma_i)^2 |p''(\tilde{\xi}_j)|} + \mathcal{O}(\epsilon^2), \text{ as } \epsilon \to 0.$$

Moreover, with $\hat{\xi}_m := 1$, then for all $0 \le j \le m$,

(55) $|p_{i,\epsilon}(\hat{\xi}_j)| = |p(\tilde{\xi}_j)| \{1 + \frac{\epsilon}{(\tilde{\xi}_j - \tilde{\sigma}_i)}\} + \mathcal{O}(\epsilon^2)$, as $\epsilon \to 0$.

Now, consider any i, $1 \le i \le m$, for which $|p(\tilde{\xi}_{i-1})| > |p(\tilde{\xi}_i)|$. From the inequalities of (51), it follows that $\tilde{\xi}_0 - \tilde{\sigma}_i < \cdots < \tilde{\xi}_{i-1} - \tilde{\sigma}_i < 0 < \tilde{\xi}_i - \tilde{\sigma}_i < \cdots < \tilde{\xi}_m - \tilde{\sigma}_i$, whence, for all $\epsilon > 0$,

(56)
$$(1 - \frac{\epsilon}{|\tilde{\xi}_{i-1} - \tilde{\sigma}_i|}) < \cdots < (1 - \frac{\epsilon}{|\tilde{\xi}_0 - \tilde{\sigma}_i|}) < 1 < (1 + \frac{\epsilon}{|\tilde{\xi}_m - \tilde{\sigma}_i|}) < \cdots < (1 + \frac{\epsilon}{|\tilde{\xi}_i - \tilde{\sigma}_i|}).$$

But with the inequalities of (50) and (56), we then have

(57)
$$\begin{cases} |p(\tilde{\xi}_{i-1})|(1 - \frac{\epsilon}{|\tilde{\xi}_{i-1} - \tilde{\sigma}_i|}) < \cdots < |p(\tilde{\xi}_0)|(1 - \frac{\epsilon}{|\tilde{\xi}_0 - \tilde{\sigma}_i|}), \text{ and} \\ \\ |p(\tilde{\xi}_m)|(1 + \frac{\epsilon}{|\tilde{\xi}_m - \tilde{\sigma}_i|}) < \cdots < |p(\tilde{\xi}_i)|(1 + \frac{\epsilon}{|\tilde{\xi}_i - \tilde{\sigma}_i|}). \end{cases}$$

But since $|p(\tilde{\xi}_{i-1})| > |p(\tilde{\xi}_i)|$, it is clear from (55) that for a suitable $\hat{\epsilon} > 0$ sufficiently small, we have $|p_{i,\hat{\epsilon}}(\hat{\xi}_{i-1})| \le |p_{i,\hat{\epsilon}}(\hat{\xi}_i)|$, that $p_{i,\hat{\epsilon}}$ again satisfies the hypotheses of Lemma 10, and that

(58) $\|p\|_{[0,1]} - |p(\tilde{\xi}_m)| > \|p_{i,\hat{\epsilon}}\|_{[0,1]} - |p_{i,\hat{\epsilon}}(\hat{\xi}_m)|$.

In addition, the i-th zero, $\tilde{\sigma}_i - \hat{\epsilon}$, of $p_{i,\hat{\epsilon}}$ is less than the corresponding zero, $\tilde{\sigma}_i$, of p, and, by virtue of the construction above, $|p_{i,\hat{\epsilon}}(\hat{\xi}_{j-1})| < |p_{i,\hat{\epsilon}}(\hat{\xi}_j)|$ for <u>all</u> $1 \le j \le m$. Thus, this iterative refinement of p can next be applied to <u>all</u> $1 \le i \le m$. It is then clear that this Remez-like iterative refinement of p can be continued indefinitely and, from the characterization in Proposition 2, must yield in the limit $Q_{s,m}$, whose i-th non-trivial zero, $\sigma_i^{(s,m)}$, is a strict lower bound for the i-th nontrivial zero of any p satisfying the hypotheses of Lemma 10, which gives (52).

Corollary 11. For any $m > 0$, denote the zeros of the Jacobi polynomial $P_m^{(-\frac{1}{2},2s-\frac{1}{2})}(t)$ by t_i, $1 \le i \le m$, where $-1 < t_1 < t_2 < \cdots < t_m < 1$. Then, (cf. (48))

(59) $\sigma_i^{(s,m)} < (t_i+1)/2$ for all $1 \le i \le m$.

Proof. The product $x^s P_m^{(-\frac{1}{2},2s-\frac{1}{2})}(2x-1)$, an element of $\pi_{s,m}$, is known from [6, Lemma 3.3] to satisfy the hypotheses of Lemma 10. Thus, (59) follows immediately from (52) of Lemma 10.

Using basically the same method of proof as in Lemma 10, we also have

Lemma 12. With the notation and hypotheses of Lemma 10, with (50) replaced by

(60) $|p(\tilde{\xi}_m)| = \|p\|_{[0,1]} \ge |p(\tilde{\xi}_{m-1})| \ge \cdots \ge |p(\tilde{\xi}_0)|$,

if $\|p\|_{[0,1]} > |p(\tilde{\xi}_0)|$, then

(61) $\tilde{\sigma}_i < \sigma_i^{(s,m)}$ for all $1 \le i \le m$.

Corollary 13. For any $m > 0$, denote the zeros of the Jacobi polynomial $P_m^{(0,2s)}(t)$ by $\tilde{t}_i$, $1 \le i \le m$, where $-1 < \tilde{t}_1 < \tilde{t}_2 < \cdots < \tilde{t}_m < 1$. Then, (cf. (48))

(62) $(\tilde{t}_i + 1)/2 < \sigma_i^{(s,m)}$ for all $1 \le i \le m$.

To illustrate the results of Corollaries 11 and 13, the nontrivial zeros of the constrained Chebyshev polynomial $T_{2,2}(x)$ have been computed to be

$$\sigma_1^{(2,2)} \doteq 0.6070 \quad ; \quad \sigma_2^{(2,2)} \doteq 0.9519.$$

The zeros of $P_2^{(-\frac{1}{2},\frac{7}{2})}((1+t)/2)$, which are upper bounds for $\sigma_i^{(2,2)}$ from Corollary 11 are

$$0.6182 \quad ; \quad 0.9532,$$

while the zeros of $P_2^{(0,4)}((1+t)/2)$, which are lower bounds for $\sigma_i^{(2,2)}$ from Corollary 13, are

$$0.5863 \quad ; \quad 0.9137.$$

Using Corollaries 11 and 13, we establish our final result:

<u>Proposition 14</u>. For each fixed θ with $0 \le \theta \le 1$, and for every infinite sequence of ordered pairs $\{(s_i,m_i)\}_{i=1}^{\infty}$ with $\lim_{i\to\infty}(s_i+m_i) = \infty$ and $\lim_{i\to\infty}\frac{s_i}{s_i+m_i} = \theta$, then the zeros of the constrained Chebyshev polynomials $\{T_{s_i,m_i}(x)\}_{i=1}^{\infty}$ are dense in $[\theta^2,1]$.

<u>Proof</u>. By means of the Sturm Comparison Theorem, it is shown in Moak, Saff, and Varga [4] that the zeros of $\{x^{s_i}P_{m_i}^{(-\frac{1}{2},2s_i-\frac{1}{2})}(2x-1)\}_{i=1}^{\infty}$ and $\{x^{s_i}P_{m_i}^{(0,2s_i)}(2x-1)\}_{i=1}^{\infty}$ are, from the hypotheses, each dense in $[\theta^2,1]$. Applying inequalities (51) and (62) shows that the zeros of $\{T_{s_i,m_i}(x)\}_{i=1}^{\infty}$ are also dense in $[\theta^2,1]$.

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E. B. Saff
Department of Mathematics
University of South Florida
Tampa, FL 33620

R. S. Varga
Department of Mathematics
Kent State University
Kent, OH 44242

Errata for "On Incomplete Polynomials" by E. B. Saff and R. S. Varga

p. 285, eq. (17). Read "for all real $x \notin (0,1)$"

p. 292, line -3. Read ", and hence, using (4),"

p. 295, line -11. Read "$\geq$" for "$\leq$".