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INCOMPLETE POLYNOMIALS: AN ELECTROSTATICS APPROACH

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Respectfully dedicated to Professor G. G. Lorentz, whose pathfinding results and whose open problems and conjectures in mathematics have greatly inspired us all.

I. INTRODUCTION

In 1976, G. G. Lorentz [4] introduced the study of certain constrained polynomials which are referred to as <u>incomplete polynomials</u>. By an <u>incomplete polynomial of type</u> θ, $0 < \theta < 1$, we mean any real or complex polynomial (of any degree) which can be written in the form

$$P(x) = \sum_{k=s}^{n} a_k x^k \text{ , where } s \geq \theta n \text{ , } s > 0. \qquad (1.1)$$

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For the most part, research interest has focused upon the behavior, relative to the interval [0, 1], of such polynomials P(x) for θ fixed. For example, on denoting the supremum norm over a set $B \subset \mathbb{C}$ by

$$\|g\|_B := \sup\{|g(z)| : z \in B\},$$

and letting

$$I_\theta := \{P : P \text{ is an incomplete polynomial of type } \theta\}, \quad 0<\theta<1, \tag{1.2}$$

we have the following fundamental property of incomplete polynomials:

Theorem 1.1. (Kemperman and Lorentz [2], Saff and Varga [6], [7]). If $P \in I_\theta$, $P \not\equiv 0$, and if $\xi \in [0, 1]$ is any point for which $|P(\xi)| = \|P\|_{[0,1]}$, then $\xi \geq \theta^2$.

It is moreover shown in [6] that $\xi > \theta^2$, and in [7] that this lower bound θ^2 is sharp in the sense that if $\xi(P)$ is the smallest such ξ in [0, 1] with $|P(\xi)| = \|P\|_{[0,1]}$ for each $P \not\equiv 0$ in I_θ, then

$$\inf\{\xi(P) : P \not\equiv 0 \text{ in } I_\theta\} = \theta^2.$$

In this regard, letting π_r denote the set of all real polynomials of degree at most r (with $\pi_{-1} := \{0\}$), consider the following extremal problem. Given any pair (s, m) of nonnegative integers, set

$$\begin{aligned} \mathcal{E}_{s,m} &:= \min\{\|x^s(x^m - g_{m-1}(x))\|_{[0,1]} : g_{m-1} \in \pi_{m-1}\} \\ &= \|x^s(x^m - \hat{g}_{m-1}(x))\|_{[0,1]}. \end{aligned} \tag{1.3}$$

Then, the incomplete polynomial

$$\mathcal{T}_{s,m}(x) := x^s(x^m - \hat{g}_{m-1}(x))/\mathcal{E}_{s,m}, \tag{1.4}$$

which is of type s/n with n:= s + m, is called the constrained Chebyshev polynomial of degree n, having a constrained zero of order s at x = 0.

Several properties of these constrained Chebyshev polynomials were obtained in [7]. In particular, we remarked there that $\mathcal{T}_{s,m}(x)$ attains its maximum absolute value on [0, 1] in precisely m + 1 points, with necessarily alternating signs. Hence, if $s/(s+m) \geq \theta$, then, by Theorem 1.1, these alternation points must lie in $(\theta^2, 1]$, and consequently, $(\theta^2, 1)$ contains all nontrivial (i.e., nonzero) zeros of $\mathcal{T}_{s,m}(x)$.

One purpose of this note is to obtain (cf. Theorem 3.6) the precise asymptotic distribution of these zeros for any sequence $\{\mathcal{T}_{s_i,m_i}(x)\}_{i=1}^{\infty}$ for which $s_i/n_i \to \theta$ and $n_i \to \infty$ (where $n_i := s_i + m_i$). Our approach, which is to study the electrostatics analogue of the problem, also provides a streamlined method for proving several of the fundamental properties of incomplete polynomials.

The outline of this paper is as follows. In Section II, we discuss a generalization of the incomplete polynomials of (1.1), and give some known results. In Section III, we state and prove our main results on the asymptotic distribution of zeros, and in Section IV, we mention two related problems.

II. POLYNOMIALS VANISHING AT BOTH ENDPOINTS

Note that an incomplete polynomial P(x) in (1.1) has a zero of order at least θn at the left endpoint of the interval [0, 1]. In [3], Lachance, Saff, and Varga studied the more general possibility of polynomials vanishing at <u>both</u> endpoints of an interval. For reasons that will be subsequently clear, we take this interval to be [-1, 1]. Then, by an <u>incomplete polynomial of type</u> (θ_1, θ_2), where $0 \leq \theta_1$, $0 \leq \theta_2$, and $0 < \theta_1 + \theta_2 < 1$, we mean any real or complex polynomial which can be written in the form

$$p(t) = (t-1)^{s_1}(t+1)^{s_2} \sum_{k=0}^{n-s_1-s_2} \alpha_k t^k, \tag{2.1}$$

where

$$s_1 \geq \theta_1 n, \ s_2 \geq \theta_2 n, \ s_1+s_2 > 0.$$

Furthermore, we set

$$I_{\theta_1,\theta_2} := \{p : p \text{ is an incomplete polynomial of type } (\theta_1,\theta_2)\}. \tag{2.2}$$

We remark that the collection I_{θ_1,θ_2} contains polynomials of arbitrarily large degree, and is closed under ordinary multiplication, but not under addition.

With the above notation, the generalization of Theorem 1.1, relative to the interval $[-1, 1]$, is as follows.

Theorem 2.1. (Lachance, Saff, and Varga [3]). *If* $p \in I_{\theta_1,\theta_2}$, $p \not\equiv 0$, *and if* $\xi \in [-1, 1]$ *is any point for which* $|p(\xi)| = \|p\|_{[-1,1]}$, *then*

$$a(\theta_1, \theta_2) \leq \xi \leq b(\theta_1, \theta_2), \tag{2.3}$$

where, with $\sigma := \theta_2 + \theta_1$, $\delta := \theta_2 - \theta_1$, a *and* b *are given by*

$$\left.\begin{aligned} a = a(\theta_1, \theta_2) &:= \sigma\delta - \sqrt{(1-\sigma^2)(1-\delta^2)} \\ b = b(\theta_1, \theta_2) &:= \sigma\delta + \sqrt{(1-\sigma^2)(1-\delta^2)}\,. \end{aligned}\right\} \tag{2.4}$$

Note that when $\theta_1 = 0$ and $\theta_2 = \theta$, we find $a = 2\theta^2 - 1$, and $b = 1$, so that the interval $2\theta^2 - 1 \leq t \leq 1$ of (2.3), after the transformation $x = (t+1)/2: [-1, 1] \to [0, 1]$, becomes $\theta^2 \leq x \leq 1$, which agrees with Theorem 1.1. Unlike Theorem 1.1, however, the sharpness of both endpoints a and b for the general case of Theorem 2.1 has not been previously established. In the next section, we prove that the interval in (2.3) *is*, in general, best possible. For this purpose, we study the properties of the <u>two-endpoint constrained Chebyshev polynomials</u> $T_{s_1,s_2,m}(t)$ which are defined as follows (cf. [3]).

Let (s_1, s_2, m) be any triple of nonnegative integers, and set

$$E_{s_1,s_2,m} := \min\{\|(t-1)^{s_1}(t+1)^{s_2}(t^m - h_{m-1}(t))\|_{[-1,+1]} : h_{m-1} \in \pi_{m-1}\}$$
$$= \|(t-1)^{s_1}(t+1)^{s_2}(t^m - \hat{h}_{m-1}(t))\|_{[-1,+1]}, \tag{2.5}$$

and

$$T_{s_1,s_2,m}(t) := (t-1)^{s_1}(t+1)^{s_2}\{t^m - \hat{h}_{m-1}(t)\}/E_{s_1,s_2,m}. \tag{2.6}$$

Note that $T_{s_1,s_2,m}$ is an incomplete polynomial of type $(s_1/n, s_2/n)$, where $n := s_1 + s_2 + m$ is its total degree.

III. MAIN RESULTS

Our approach to describing the behavior of incomplete polynomials of type (θ_1, θ_2), relative to $[-1, 1]$, is to consider the following

<u>Electrostatics Problem</u>. Let $0 \le \theta_1$, $0 \le \theta_2$, satisfy $0 < \theta_1 + \theta_2 < 1$. Suppose that on the interval $[-1, 1]$, a fixed charge of amount θ_1 is placed at $t = 1$, a fixed charge of amount θ_2 is placed at $t = -1$, and a continuous charge of amount $1 - \theta_1 - \theta_2$ is placed on $[-1, 1]$, allowing it to reach equilibrium, the only constraint being that the charges remain confined to the interval $[-1, 1]$. Here and throughout, the logarithmic potential and its corresponding force is assumed. Then, the problem is to describe the distribution of the continuous charge.

This is answered in

<u>Theorem 3.1</u>. <u>For the electrostatics problem described above, the continuous charge of amount</u> $1 - \theta_1 - \theta_2$ <u>lies entirely in the interval</u>

$a(\theta_1, \theta_2) \le t \le b(\theta_1, \theta_2)$, where a and b are defined in (2.4). Moreover, the point density of the charge $1 - \theta_1 - \theta_2$ on this interval is given by

$$\frac{\sqrt{(t-a)(b-t)}}{\pi(1-t^2)}, \quad a \le t \le b. \tag{3.1}$$

We shall prove Theorem 3.1 by considering the limit of the corresponding problem for discrete point charges.

Lemma 3.2. For each sufficiently large integer n, let $s_1(n)$, $s_2(n)$ be positive integers such that $s_1(n) + s_2(n) < n$ and such that

$$\lim_{n\to\infty} \frac{s_1(n)}{n} = \theta_1, \ \lim_{n\to\infty} \frac{s_2(n)}{n} = \theta_2, \qquad \theta_1 + \theta_2 < 1. \tag{3.2}$$

Suppose that a charge of amount $s_1(n)/n$ is placed at $t = 1$, and a charge of $s_2(n)/n$ is placed at $t = -1$, and let $m(n) := n - s_1(n) - s_2(n)$ point charges, each of amount $1/n$, be placed on $(-1, 1)$ so that equilibrium is reached. Let $-1 < t_{n,1} < t_{n,2} < \cdots < t_{n,m(n)} < 1$ denote the location of these point charges, and set

$$f_{m(n)}(z) := \prod_{k=1}^{m(n)} (z - t_{n,k}). \tag{3.3}$$

Then, uniformly on each closed set in $\mathbb{C}\setminus[a(\theta_1, \theta_2), b(\theta_1, \theta_2)]$, we have

$$\lim_{n\to\infty} \frac{1}{m(n)} \frac{f'_{m(n)}(z)}{f_{m(n)}(z)} = \psi(z), \tag{3.4}$$

where

$$\psi(z) := \frac{\delta - \sigma z + \sqrt{(z-a)(z-b)}}{(1-\sigma)(z^2-1)}, \tag{3.5}$$

the quantities σ, δ, a, and b being defined in (2.4).

We remark that, in (3.5), $\sqrt{(z-a)(z-b)}$ denotes the branch with cut $[a, b]$ that behaves like $+z$ near ∞.

Proof. From the equilibrium equations

$$\frac{1}{n}\left[\sum_{\substack{j=1\\ j\ne k}}^{m(n)} \frac{1}{t_{n,k}-t_{n,j}} + \frac{s_1(n)}{t_{n,k}-1} + \frac{s_2(n)}{t_{n,k}+1}\right] = 0, \ k = 1, \cdots, m(n),$$

it follows, as in Szegö [10, p. 141], that $f_{m(n)}(z)$ satisfies the differential equation

$$(z^2-1)f''_{m(n)} + 2\{[s_1(n) + s_2(n)]z + s_1(n) - s_2(n)\}f'_{m(n)}$$
$$= \{m(n)[m(n)-1]+2m(n)[s_1(n) + s_2(n)]\}f_{m(n)}. \tag{3.6}$$

In fact, $f_{m(n)}(z)$ is just a constant multiple of the Jacobi polynomial $P^{(\alpha,\beta)}_{m(n)}(z)$, where $\alpha := 2s_1(n) - 1$ and $\beta := 2s_2(n) - 1$ (cf. [10, p. 140]). Now, with the assumption of (3.2), it is proved in [5] that the sequence of these Jacobi polynomials, as $n \to \infty$, has no limit point of zeros exterior to $[a, b]$. Hence, the functions

$$\psi_n(z) := \frac{1}{m(n)} \frac{f'_{m(n)}(z)}{f_{m(n)}(z)} \tag{3.7}$$

form a normal family of analytic functions in the complement of $[a, b]$, relative to the extended complex plane $\mathbb{C}^*$.

On dividing (3.6) by $f_{m(n)}$ and using the easily verified relations

$$\frac{f'_{m(n)}(z)}{f_{m(n)}(z)} = m(n)\psi_n(z)\ , \quad \frac{f''_{m(n)}(z)}{f_{m(n)}(z)} = m(n)\psi'_n(z) + m^2(n)\psi_n^2(z),$$

we find that $\psi_n(z)$ of (3.7) satisfies

$$(z^2 - 1)[m(n)\psi'_n + m^2(n)\psi_n^2] + 2\{[s_1 + s_2]z + s_1 - s_2\}m(n)\psi_n$$
$$= m(n)[m(n) - 1] + 2m(n)(s_1 + s_2), \tag{3.8}$$

where, for convenience, we have written s_1 for $s_1(n)$, s_2 for $s_2(n)$. Now, let $\psi(z)$ be any limit function (as $n \to \infty$) in $\mathbb{C}^*\setminus[a, b]$ of any subsequence of the normal family $\{\psi_n(z)\}$. Since $\psi'(z)$ is the limit of the corresponding sequence of derivatives, it follows from (3.8), on dividing by n^2 and using the assumptions (3.2), that $\psi(z)$ satisfies the quadratic equation

$$\psi^2(z) - \frac{2(\delta-\sigma z)}{(1-\sigma)(z^2-1)}\psi(z) - \frac{1+\sigma}{(1-\sigma)(z^2-1)} = 0, \tag{3.9}$$

where $\sigma = \theta_2 + \theta_1$, $\delta = \theta_2 - \theta_1$. Hence,

$$\psi(z) = \frac{\delta - \sigma z \pm \sqrt{(z-a)(z-b)}}{(1-\sigma)(z^2-1)}. \tag{3.10}$$

Since $z\psi_n(z) \to 1$ as $z \to \infty$ for each n, then necessarily also $z\psi(z) \to 1$ as $z \to \infty$, which implies that the plus sign must be taken for the radical in (3.10). Hence, $\psi(z)$ is given by (3.5).

Finally, as $\psi(z)$ represents an arbitrary limit function of any subsequence of $\{\psi_n(z)\}$ in $\mathbb{C}^*\setminus[a, b]$, the conclusion (3.4) of the lemma follows. ∎

Concerning the limiting distribution of the point charges in Lemma 3.2, we prove

<u>Lemma 3.3</u>. Let $q_m(z) = \prod_{k=1}^{m}(z - \tau_{m,k})$ be a sequence of polynomials, each having all its zeros on $[-1, 1]$, and suppose that

$$\lim_{m\to\infty} \frac{1}{m}\frac{q_m'(z)}{q_m(z)} = \psi(z), \text{ for } z \in \mathbb{C}\setminus[-1, 1], \tag{3.11}$$

where $\psi(z)$ is defined in (3.5) with $0 \le \theta_1 + \theta_2 < 1$. For each m, let ν_m denote the atomic measure (on the Borel sets of $[-1, 1]$) having mass $1/m$ each point $\tau_{m,k}$, $k = 1, 2, \cdots, m$. Then, there exists a measure ν^* such that $\nu_m \to \nu^*$ weakly as $m \to \infty$, where the support of the measure ν^* is exactly $[a(\theta_1, \theta_2), b(\theta_1, \theta_2)]$ (cf. (2.4)), and where

$$\nu^*\{(\alpha, \beta)\} = \frac{1}{\pi(1-\theta_1-\theta_2)}\int_{\alpha}^{\beta}\frac{\sqrt{(t-a)(b-t)}}{1-t^2}\,dt, \ \forall(\alpha,\beta)\subset[a,b]. \tag{3.12}$$

<u>Proof</u>. We first note that

$$\frac{1}{m}\frac{q_m'(z)}{q_m(z)} = \int_{-1}^{+1}\frac{d\nu_m(t)}{z-t}, \ \forall z \notin [-1, 1]. \tag{3.13}$$

Since $\nu_m\{[-1, 1]\} = 1$ for all m, it follows from Helly's theorem (cf. [9], [12]) that there exists a subsequence ν_{m_i} and a measure ν^* such that $\nu_{m_i} \to \nu^*$ weakly. Hence,

$$\lim_{i\to\infty} \int_{-1}^{+1} \frac{d\nu_{m_i}(t)}{z-t} = \int_{-1}^{+1} \frac{d\nu^*(t)}{z-t}, \ \forall\ z \notin [-1, 1].$$

But, from (3.13) and (3.11), this means that

$$\int_{-1}^{+1} \frac{d\nu^*(t)}{z-t} = \psi(z), \ \forall\ z \notin [-1, 1]. \tag{3.14}$$

Now, by applying the Stieltjes inversion formula (cf. [11, p. 250]) to (3.14), we have for each interval $(\alpha, \beta) \subset [-1, 1]$ that

$$\nu^*\{(\alpha,\beta)\} + \frac{\nu^*\{\alpha\}}{2} + \frac{\nu^*\{\beta\}}{2} = \lim_{y\to 0^+} \frac{-1}{\pi} \int_{\alpha}^{\beta} \operatorname{Im} \psi(t+iy)dt. \tag{3.15}$$

From (3.5), we see on the other hand that

$$\hat{\psi}(t) := \lim_{y\to 0^+} \operatorname{Im} \psi(t+iy) = \begin{cases} 0, & \text{for } t \notin [a, b] \\ \dfrac{\sqrt{(t-a)(b-t)}}{(1-\sigma)(t^2-1)}, & \text{for } t \in (a, b). \end{cases}$$

Thus, (3.15) implies that the support of ν^* is exactly the interval $[a, b]$, and since $\hat{\psi}(t)$ is continuous on $(-1, 1)$, ν^* is absolutely continuous with respect to Lebesgue measure. Consequently, (3.15) yields

$$\nu^*\{(\alpha, \beta)\} = \frac{1}{\pi(1-\sigma)} \int_{\alpha}^{\beta} \frac{\sqrt{(t-a)(b-t)}}{1-t^2}\, dt, \ \forall\ (\alpha, \beta) \subset [a, b].$$

Since this is true for every weak limit ν^*, Lemma 3.3 is proved. ∎

Proof of Theorem 3.1. Combining Lemmas 3.2 and 3.3, we see that the limit, as $n \to \infty$, of the discrete electrostatics problems described in Lemma 3.2, has a charge of θ_1 at $t = 1$, a charge of θ_2 at $t = -1$, and a charge of $1 - \theta_1 - \theta_2$ distributed continuously over the interval

$[a(\theta_1, \theta_2), b(\theta_1, \theta_2)]$, with density (cf. (3.12)) given by

$$(1 - \theta_1 - \theta_2)d\nu^*(t) = \frac{\sqrt{(t-a)(b-t)}}{\pi(1 - t^2)} dt , \quad t \in [a, b]. \quad \blacksquare$$

We remark that the function

$$V(z; \theta_1, \theta_2) := \theta_1 \ln|z-1| + \theta_2 \ln|z+1| + (1 - \theta_1 - \theta_2) \int_a^b \ln|z-t| d\nu^*(t) \qquad (3.16)$$

gives the potential for the electrostatics problem of Theorem 3.1. In fact, it can be shown that

$$V(z; \theta_1, \theta_2) \equiv \ln \Delta + \ln G(z; \theta_1, \theta_2), \qquad (3.17)$$

where

$$\Delta := \frac{1}{2} \sqrt{(1+\sigma)^{1+\sigma} (1-\sigma)^{1-\sigma} (1+\delta)^{1+\delta} (1-\delta)^{1-\delta}}, \qquad (3.18)$$

and (cf. [3, p. 425])

$$G(z; \theta_1, \theta_2) := |\varphi(z)| \left|\frac{\varphi(z)-\varphi(1)}{\varphi(1)\varphi(z)-1}\right|^{\theta_1} \left|\frac{\varphi(z)-\varphi(-1)}{\varphi(-1)\varphi(z)-1}\right|^{\theta_2}, \qquad (3.19)$$

where

$$w = \varphi(z) := (\sqrt{z-a} + \sqrt{z-b})/(\sqrt{z-a} - \sqrt{z-b})$$

maps $\mathbb{C}^*\setminus[a(\theta_1, \theta_2), b(\theta_1, \theta_2)]$ onto the exterior of the unit circle $|w| = 1$. To prove (3.17), one need only verify that the difference $V - (\ln \Delta + \ln G)$ is harmonic in $\mathbb{C}^*\setminus[a, b]$, equal to zero at ∞, and approaches a constant as z approaches the boundary segment $[a, b]$. Thus, this harmonic function is identically zero in $\mathbb{C}^*\setminus[a, b]$.

Now in [3, p. 434], it is proved that any sequence of two-endpoint constrained Chebyshev polynomials (cf. (2.6)) $T_{s_1(n),s_2(n),m(n)}(t)$ of respective degrees n for which

$$s_1(n)/n \to \theta_1, \quad s_2(n)/n \to \theta_2, \quad n := s_1(n)+s_2(n)+m(n) \to \infty, \qquad (3.20)$$

satisfies

$$\lim_{n\to\infty} |T_{s_1(n),s_2(n),m(n)}(z)|^{1/n} = G(z;\ \theta_1,\ \theta_2),\ z \in \mathbb{C}\setminus[a,\ b]. \qquad (3.21)$$

Hence, we claim that the sequence of polynomials

$$q_{m(n)}(z) := T_{s_1(n),s_2(n),m(n)}(z)/(z-1)^{s_1(n)}(z+1)^{s_2(n)},\ n = 1, 2, \cdots, \qquad (3.22)$$

satisfies the hypotheses of Lemma 3.3. Indeed, on writing $T_n(z)$ for $T_{s_1(n),s_2(n),m(n)}(z)$ and $V(z)$ for $V(z;\ \theta_1,\ \theta_2)$ for simplicity, (3.17) and (3.21) imply that

$$\frac{1}{n}\frac{T_n'(z)}{T_n(z)} \to V_x(z) - i\ V_y(z),\quad z = x+iy \in \mathbb{C}\setminus[a,\ b].$$

Hence, from (3.16) and (3.14), it follows that

$$\frac{1}{m(n)}\frac{q_{m(n)}'(z)}{q_{m(n)}(z)} \to \frac{1}{(1-\theta_1-\theta_2)}\left\{V_x(z) - i\ V_y(z) - \frac{\theta_1}{z-1} - \frac{\theta_2}{z+1}\right\} = \int_a^b \frac{d\nu^*(t)}{z-t} = \psi(z)$$

for all $z = x+iy \in \mathbb{C}\setminus[a,\ b]$. Consequently, we have

Theorem 3.4. *Let $\hat{p}_n(t) := T_{s_1(n),s_2(n),m(n)}(t)$ be any sequence of two-endpoint constrained Chebyshev polynomials of respective degrees n for which (3.20) holds. Then, the zeros of the sequence $\hat{p}_n(t)$, other than those at $t = \pm 1$, have no limit points exterior to $[a,\ b]$, and are dense in $[a,\ b]$, where a and b are defined in (2.4). Furthermore, if $N_n(\alpha,\ \beta)$ denotes the total number of zeros of $\hat{p}_n(t)$ in any interval $(\alpha,\beta) \subset [a,b]$, then*

$$\lim_{n\to\infty} \frac{N_n(\alpha,\ \beta)}{n} = \frac{1}{\pi}\int_\alpha^\beta \frac{\sqrt{(t-a)(b-t)}}{1-t^2}\,dt. \qquad (3.23)$$

Theorem 3.4 immediately implies the sharpness of Theorem 2.1. Indeed, if we consider any sequence of constrained Chebyshev polynomials from I_{θ_1,θ_2} for which (3.20) holds, then their respective alternation points all lie on $[a,\ b]$ and are interlaced by their respective zeros. As these zeros are, from Theorem 3.4, dense in $[a,\ b]$, we have

Corollary 3.5. The interval $[a, b]$ of Theorem 2.1 is sharp in the sense that, for any pair (θ_1, θ_2) with $0 \le \theta_1$, $0 \le \theta_2$, and $0 < \theta_1 + \theta_2 < 1$, there exists a sequence $\{p_n(t)\} \subset I_{\theta_1,\theta_2}$ of nonconstant polynomials and sequences of points $\xi_{n,1}$, $\xi_{n,2}$ in $[-1, 1]$ such that $|p_n(\xi_{n,1})| = |p_n(\xi_{n,2})| = \|p_n\|_{[-1,1]}$ with $\lim_{n\to\infty} \xi_{n,1} = a$; $\lim_{n\to\infty} \xi_{n,2} = b$.

We conclude this section by describing the distribution of zeros for the (one-endpoint) constrained Chebyshev polynomials of (1.4). This is just a special case of Theorem 3.4 when $\theta_1 = 0$, $\theta_2 = \theta$, and the interval $[-1, 1]$ is mapped to $[0, 1]$.

Theorem 3.6. Let $\hat{P}_n(x) := \mathcal{T}_{s(n),m(n)}(x)$ be any sequence of constrained Chebyshev polynomials (cf. (1.4)) of respective degrees $n = s(n)+m(n)$ for which $s(n)/n \to \theta$ as $n \to \infty$, where $0 < \theta < 1$. Then, the zeros of $\hat{P}_n(x)$, other than those at $x = 0$, have no limit point exterior to $[\theta^2, 1]$, and are dense in $[\theta^2, 1]$. Furthermore, if $\mathfrak{N}_n(\alpha, \beta)$ denotes the total number of zeros of $\hat{P}_n(x)$ in any interval $(\alpha, \beta) \subset [\theta^2, 1]$, then

$$\lim_{n\to\infty} \frac{\mathfrak{N}_n(\alpha,\beta)}{n} = \frac{1}{\pi}\int_{\alpha}^{\beta} \frac{1}{x}\sqrt{\frac{x-\theta^2}{1-x}}\,dx. \tag{3.24}$$

IV. RELATED QUESTIONS

Concerning the possibility of uniform approximation of continuous functions on $[0, 1]$ by incomplete polynomials of type θ (cf. (1.1)), the following result is known.

Theorem 4.1. (Saff and Varga [8], v. Golitschek [1]). Let $0 < \theta < 1$ be fixed, and let $F \in C[0, 1]$ with $F \notin I_\theta$ (cf. (1.2)). Then, a necessary and sufficient condition that F be the uniform limit on $[0, 1]$ of a sequence of incomplete polynomials of type θ is that $F(x) = 0$ for all $0 \le x \le \theta^2$.

For the more general two-endpoint case, the characterization of the uniform limits on $[-1, 1]$ of incomplete polynomials of type (θ_1, θ_2) remains an open question. However, the results of our investigations suggest the following

Conjecture 4.2. Let $0 < \theta_1$, $0 < \theta_2$, with $0 < \theta_1 + \theta_2 < 1$, and let $F \in C[-1, 1]$ with $F \notin I_{\theta_1,\theta_2}$ (cf. (2.2)). Then, a necessary and sufficient condition that F be the uniform limit on $[-1, 1]$ of a sequence of incomplete polynomials of type (θ_1, θ_2) is that $F(t) = 0$ for all $t \in [-1, a(\theta_1, \theta_2)] \cup [b(\theta_1, \theta_2), 1]$, where a and b are given in (2.4).

Another related question concerns the behavior of incomplete polynomials that have a high-order zero at an interior point λ of $[0, 1]$, namely, the collection

$$I_\theta^{(\lambda)} := \{P: P(x) = \sum_{k=s}^{n} a_k (x-\lambda)^k, \ s \geq \theta n, \ s > 0\}, \ 0 < \lambda < 1. \tag{4.1}$$

Here, the related electrostatics problem involves placing a fixed charge of amount θ at $x = \lambda$, and the remaining charge of amount $1 - \theta$ on $[0, 1]$, so that equilibrium is reached. It seems likely that the methods employed in this paper can be adapted to find the distribution of such charges, part of which will lie on $[0, x_1(\lambda)]$ and part on $[x_2(\lambda), 1]$, where $x_1(\lambda) < \lambda < x_2(\lambda)$. However, the analysis is more involved in that elliptic integrals arise. This generalization will be reserved for a later occasion.

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