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AN EXTENSION OF THE ENESTRÖM-KAKEYA THEOREM AND ITS SHARPNESS*

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Abstract. The classical Eneström-Kakeya Theorem, for obtaining bounds for the moduli of the zeros of any polynomial with positive coefficients, is extended to the case of any complex polynomial having no zeros on the ray $[0, +\infty)$. It is shown that this extension is sharp in the sense that, given such a complex polynomial $p_n(z)$ of degree $n \geqq 1$, a sequence of polynomials $\{Q_{m_i}(z)\}_{i=1}^{\infty}$ can be found for which the classical Eneström-Kakeya Theorem, applied to the products $Q_{m_i}(z)p_n(z)$, yields, in the limit as $i \to \infty$, the maximum of the moduli of the zeros of $p_n(z)$.

A computational algorithm, based on linear programming, is also described whereby nearly "optimal" multiplying polynomials $Q_{m_i}(z)$ can be computed.

1. Introduction. With π_n denoting the set of all complex polynomials of degree exactly n, and with

$$\pi_n^+ := \{p_n(z) = \sum_{j=0}^{n} a_j z^j : a_j > 0 \text{ for all } j = 0, 1, \cdots, n\}, \tag{1.1}$$

a useful form of the classical Eneström-Kakeya Theorem [4], [13], due in fact to Eneström [4], is the following:

THEOREM A. *For any* $p_n(z) = \sum_{j=0}^{n} a_j z^j$ *in* π_n^+ *with* $n \geqq 1$, *define*

$$\alpha = \alpha[p_n] := \min_{0 \leqq i < n} \left\{ \frac{a_i}{a_{i+1}} \right\}, \qquad \beta = \beta[p_n] := \max_{0 \leqq i < n} \left\{ \frac{a_i}{a_{i+1}} \right\}. \tag{1.2}$$

Then, all the zeros of $p_n(z)$ *lie in the annulus*

$$\alpha \leqq |z| \leqq \beta. \tag{1.3}$$

Evidently, if

$$\rho(p_n) := \max \{|z_j| : p_n(z_j) = 0\} \tag{1.4}$$

denotes the *spectral radius* of any complex polynomial $p_n(z)$ of degree at least unity, then it follows from (1.3) of Theorem A that

$$\beta[p_n] \geqq \rho(p_n) \quad \forall p_n(z) \in \pi_n^+, \quad \forall n \geqq 1. \tag{1.5}$$

Naturally, it is of interest to know when the inequality of (1.5) is sharp. This was first studied by Hurwitz [11], and the following result of [1] is a corrected form of Hurwitz's original contribution. (A similar result can be analogously obtained for the sharpness of $\alpha[p_n]$ in estimating the minimum of the moduli of the zeros of $p_n(z)$; see [1].)

THEOREM B. *For any* $p_n(z) = \sum_{j=0}^{n} a_j z^j$ *in* π_n^+ *with* $n \geqq 1$, *define*

$$\bar{S} = \bar{S}[p_n] := \{j = 1, 2, \cdots, n+1 : \beta a_{n+1-j} - a_{n-j} > 0\}, \quad \textit{where } a_{-1} := 0, \tag{1.6}$$

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and

(1.7) $$\bar{k} = \bar{k}[p_n] := \text{g.c.d.}\,\{j \in \bar{S}\}.$$

Then, equality in (1.5) *is valid iff* $\bar{k} > 1$. *If* $\bar{k} > 1$, *the zeros of* $p_n(z)$ *on* $|z| = \beta$ *are all simple, and are precisely given by*

(1.8) $$\beta \exp\{2\pi i j/\bar{k} : j = 1, 2, \cdots, \bar{k} - 1\}.$$

Moreover, $p_n(z)$ *has the form*

(1.9) $$p_n(\beta z) = \{1 + z + z^2 + \cdots + z^{\bar{k}-1}\} q_m(z^{\bar{k}}),$$

where $q_m \in \pi_m^+$. *If* $m \geqq 1$, *all the zeros of* $q_m(w)$ *lie in* $|w| < 1$, *and* $\beta[q_m] \leqq 1$.

Now, the Eneström-Kakeya upper bound $\beta[p_n]$ for $\rho(p_n)$ from (1.5) is certainly an easy quantity to compute. But, it suffers from two serious deficiencies. First, this upper bound can be applied only to the rather limited set of polynomials $\bigcup_{n=1}^{\infty} \pi_n^+$. For example, it cannot be applied as such to the particular polynomial $f_1(z) = 1 + z^2$. Second, the upper bound $\beta[p_n]$ may be a poor estimate of $\rho(p_n)$, and it is not apparent how this situation can be improved. For example, if $f_2(z) = 1 + \varepsilon z + z^2$ where $0 < \varepsilon \leqq 1$, we find that $\beta[f_2] = \varepsilon^{-1}$, which is a crude upper bound for $\rho(f_2) = 1$, when ε is small.

To explain our approach of generalizing the Eneström-Kakeya Theorem, note in the first example above that if $Q_1(z) = 1 + z$, then the product $Q_1(z) \cdot f_1(z) = 1 + z + z^2 + z^3$ is an element of π_3^+. On applying Theorem A, we obtain that $\beta[Q_1 \cdot f_1] = 1 \geqq \rho(Q_1 f_1)$. Moreover, since $\rho(Q_1 f_1) \geqq \rho(f_1)$ from (1.4), then

$$1 = \beta[Q_1 f_1] \geqq \rho(f_1),$$

and this last inequality is sharp since $\rho(f_1) = 1$. Similarly, for the second example above we find that

$$\beta[Q_1 f_2] = 1 + \varepsilon > \rho(f_2) = 1,$$

this upper bound being a sharper estimate of $\rho(f_2) = 1$ than the classical Eneström-Kakeya bound ε^{-1}, when ε is small.

More generally, for some complex polynomial $p_n(z)$ in π_n with $n \geqq 1$, suppose that there is a nonnegative integer m and a multiplier polynomial $Q_m(z)$ in π_m such that $Q_m(z) \cdot p_n(z) \in \pi_{n+m}^+$. Then, on applying (1.5), we have $\beta[Q_m p_n] \geqq \rho(Q_m p_n) \geqq \rho(p_n)$, i.e.,

(1.10) $$\beta[Q_m p_n] \geqq \rho(p_n),$$

and we call $\beta[Q_m p_n]$ a *generalized Eneström-Kakeya functional* for $p_n(z)$.

Several questions now arise, the first being to find the precise class of polynomials $p_n(z)$ for which the generalized Eneström-Kakeya functional is defined. This is answered in

PROPOSITION 1. *Given* $p_n(z) \in \pi_n$ *with* $n \geqq 1$, *there exists a nonnegative integer* m *and a* $Q_m(z) \in \pi_m$ *for which* $Q_m(z) \cdot p_n(z) \in \pi_{m+n}^+$ *iff* $p_n(z)$ *has no zeros on the ray* $[0, +\infty)$.

The proof of this result will be given in § 3. Because of Proposition 1, it is convenient then to set

(1.11) $$\hat{\pi}_n := \{p_n(z) \in \pi_n : p_n(z) \text{ has no zeros on the ray } [0, +\infty)\} \quad \text{for } n \geqq 1.$$

The next results, aimed at the sharpness of the inequality of (1.10), are our main results. Their proofs are given in §§ 4 and 5.

THEOREM 1. *For each $p_n(z) \in \hat{\pi}_n$ with $n \geqq 1$, there exists a sequence of polynomials $\{Q_{m_i}(z)\}_{i=1}^{\infty}$, with $Q_{m_i}(z) \in \pi_{m_i}$ and with $Q_{m_i}(z) \cdot p_n(z) \in \overset{+}{\pi}_{m_i+n}$ for all $i \geqq 1$, such that*

$$\lim_{i \to \infty} \beta[Q_{m_i} p_n] = \rho(p_n). \tag{1.12}$$

In essence, Theorem 1 gives us that the generalized Eneström-Kakeya functional is asymptotically *sharp* in the sense of (1.12).

Another question that can be asked is to characterize those elements $p_n \in \hat{\pi}_n$ with $n \geqq 1$ for which equality holds in (1.10) for *some* polynomial $Q_m(z)$, as opposed to equality holding in the limit as in (1.12) of Theorem 1. This is answered in

THEOREM 2. *Given $p_n(z) \in \hat{\pi}_n$ with $n \geqq 1$, there exists a nonnegative integer m and a polynomial $Q_m(z)$ in π_m with $Q_m(z) \cdot p_n(z) \in \overset{+}{\pi}_{m+n}$, such that*

$$\beta[Q_m p_n] = \rho(p_n) \tag{1.13}$$

iff all of the following hold:

(1.14)
- (i) *All zeros of $p_n(z)$ of modulus $\rho(p_n)$ are simple.*
- (ii) *If $\{\zeta_j\}_{j=1}^{r}$ denotes the set of all zeros of $p_n(z)$ on the circle $|z| = \rho(p_n)$, then $\arg \zeta_j$ is a (nonzero) rational multiple of 2π, i.e., $\arg \zeta_j = 2\pi n_j/d_j$ (in lowest terms), where n_j and d_j are positive integers with $0 < n_j < d_j$ for all $j = 1, 2, \cdots, r$.*
- (iii) *If $D := \text{l.c.m.}\ \{d_j\}_{j=1}^{r}$, there is a positive integer σ such that, for every zero ζ of $p_n(z)$ with $|\zeta| < \rho(p_n)$; we have $\zeta^{\sigma D} \notin [0, +\infty)$.*

It is interesting to note that the motivation for Theorems 1 and 2 comes directly from Theorem B, in the sense that the polynomial $p_n(\beta z)$ of (1.9) of Theorem B is such that its zeros have a *ring-like character*; i.e., $p_n(\beta z)$ has $\bar{k} - 1$ zeros nearly uniformly distributed on $|z| = 1$, while its remaining zeros are distributed as the $\bar{k}$th roots of zeros of $q_m(w)$ (cf. (1.9)). This pattern persists, as we shall see, both in our examples as well as in the spirit of the proofs of Theorems 1 and 2.

In the next section, we show how linear programming techniques can be used to determine nearly "optimal" polynomial multipliers $Q_m(z)$ of a specific degree such that $Q_m(z) \cdot p_n(z) \in \overset{+}{\pi}_{m+n}$. In addition, the results of some numerical experiments will be given and discussed.

Because of the continuing interest in the classical Eneström-Kakeya Theorem and its many generalizations, we have gathered in the References a number of books and papers which deal in part with this topic, in the hope that such a list may be of value to the readers.

2. Optimization of the generalized Eneström-Kakeya functional. For any $p_n \in \hat{\pi}_n$, set

$$\omega_m(p_n) := \{Q_m(z) \in \pi_m : Q_m(z) \cdot p_n(z) \in \overset{+}{\pi}_{m+n}\} \quad \text{for any } m \geqq 0. \tag{2.1}$$

Note that $\omega_m(p_n)$ may be empty for a particular nonnegative integer m, but from Proposition 1, it follows there is a nonnegative integer m_0 such that $\omega_{m_0}(p_n) \neq \varnothing$. As is easily seen, $\omega_m(p_n) \neq \varnothing$ implies $\omega_{m+k}(p_n) \neq \varnothing$ for every $k \geqq 1$. Thus, as a consequence of Proposition 1, there is a *least* nonnegative integer $\sigma(p_n)$ for each $p_n \in \hat{\pi}_n$ such that

$$\omega_m(p_n) \neq \varnothing, \quad \text{for all } m \geqq \sigma(p_n). \tag{2.2}$$

Note also that $\omega_m(p_n) \neq \varnothing$ implies that $\omega_m(p_n)$ is a *convex* subset of π_m; i.e., if $q_1(z)$ and $q_2(z)$ are in $\omega_m(p_n)$, then so is $\alpha q_1(z) + (1 - \alpha) q_2(z)$ for all $0 \leqq \alpha \leqq 1$.

Given a $p_n \in \hat{\pi}_n$, and given that $\omega_m(p_n) \neq \varnothing$, it is of interest to determine computationally a nearly "optimum" element $\tilde{Q}_m(z)$ in $\omega_m(p_n)$, i.e., one whose generalized Eneström-Kakeya functional satisfies:

$$\beta[\tilde{Q}_m p_n] \doteqdot \inf \{\beta[Q_m p_n]: Q_m \in \omega_m(p_n)\}.$$

This can be done by solving a sequence of linear programming subproblems, each of which consists of finding a so-called feasible solution [17, § 3.5] to a set of linear inequalities. (Such computational subproblems are usually solved using "Phase I" of the simplex method; see [17].) Specifically, for any fixed $p_n(z) = \sum_{j=0}^{n} a_j z^j$ in $\hat{\pi}_n$ (which we may take, without loss of generality, to be real), assume $\omega_m(p_n) \neq \varnothing$, and consider any real $Q_m(z) = \sum_{j=0}^{m} b_j z^j$. If we set

$$Q_m(z) \cdot p_n(z) = \sum_{j=0}^{m+n} \gamma_j z^j,$$

then $Q_m \in \omega_m(p_n)$ iff

(2.3) $$\gamma_j > 0 \quad \text{for all } j = 0, 1, \cdots, m+n,$$

which is a system of linear inequalities in the b_j's, since $\gamma_j = \sum_{i=\max(0;j-m)}^{\min(j;n)} a_i b_{j-i}$, for $j = 0, 1, \cdots, m+n$. We then say that $(\tau, Q_m(z))$ is a *feasible point* for $\omega_m(p_n)$ (cf. Luenberger [17, p. 18]) if, in addition to (2.3),

(2.4) $$\gamma_j \leqq \tau \gamma_{j+1} \quad \text{for all } j = 0, 1, \cdots, m+n-1.$$

By definition, if $(\tau, Q_m(z))$ is a feasible point for $\omega_m(p_n)$, then $Q_m \in \omega_m(p_n)$ and $\beta[Q_m p_n] \leqq \tau$. Note that Theorem 1 implies that given any $\tau > \rho(p_n)$, a feasible point $(\tau, Q_m(z))$ is guaranteed to exist for m sufficiently large.

On the other hand, fixing m and given a feasible point $(\tau, Q_m(z))$ for $\omega_m(p_n)$ we can proceed (see below) to determine computationally a *least* feasible point $(\tau_m, \hat{Q}_m(z))$ in $\omega_m(p_n)$, where

(2.5) $$\tau_m := \inf \{\tau: (\tau, Q_m(z)) \text{ is a feasible point in } \omega_m(p_n) \text{ for some } Q_m(z) \in \pi_m\}.$$

Note that since $(\beta[Q_m p_n], Q_m(z))$ is, by definition, a feasible point for $\omega_m(p_n)$ for each $Q_m(z) \in \omega_m(p_m)$, it follows from (2.4) and (2.5) that

(2.6) $$\tau_m = \inf \{\beta[Q_m p_n]: Q_m(z) \in \omega_m(p_n)\}.$$

Thus, our computational technique finds in essence an "optimal" multiplier polynomial in $\omega_m(p_n)$, if $\omega_m(p_n) \neq \varnothing$.

The computational experiments were carried out as follows. Given a $p_n(z) \in \hat{\pi}_n$,

1. Compute a $\tau^{(0)} > \rho(p_n)$ from the coefficients of p_n using some standard upper bound for $\rho(p_n)$ (see, e.g., [18]).

2. For $m := 1, 2, 3, \cdots$, use linear programming to attempt to find a feasible point $(\tau^{(0)}, Q_m(z))$. Call the first m for which success occurs m_0.

3. For $m := m_0, m_0+1, \cdots$, find "optimal" multipliers $\tilde{Q}_m(z)$, for each fixed m, by using a bisection technique on the variable τ, with (2.3) and (2.4) holding. For example, given $(\tau^{(0)}, Q_{m_0}(z))$, try to find a feasible point for $\tau := \tau^{(0)}/2$; if this is not possible, try with $\tau := \frac{3}{4} \cdot \tau^{(0)}$; otherwise try $\tau := \tau^{(0)}/4$, etc.

In our computations, the actual testing for feasibility ("Phase I" of the simplex method) was done using the program in Wilkinson and Reinsch [25, p. 152].

Now, let m_i be the sequence of integers and $Q_{m_i}(z)$ the sequence of polynomials in Theorem 1. By (2.5), we can compute a sequence of polynomials $\hat{Q}_{m_i}$ satisfying

(2.7) $$\beta(\hat{Q}_{m_i} p_n) = \tau_{m_i} + \varepsilon_i,$$

where the ε_i are positive quantities which can be chosen to satisfy $\lim_{i\to\infty} \varepsilon_i = 0$. But,

$$\rho(p_n) \leqq \beta(\hat{Q}_{m_i} p_n) \leqq \beta(Q_{m_i} p_n) + \varepsilon_i,$$

since from (2.6), $\tau_{m_i} \leqq \beta(Q_{m_i} p_n)$. Thus, taking limits and using (1.12),

$$\lim_{i\to\infty} \beta(\hat{Q}_{m_i} p_n) = \rho(p_n), \tag{2.8}$$

and hence (ignoring roundoff), the sequence of estimates provided by the computational algorithm is guaranteed to converge to $\rho(p_n)$.

Example 1. $p_6(z) = (z^3+1)^2 \in \hat{\pi}_6$.

For this polynomial, an optimum multiplier polynomial $\hat{Q}_{32}(z)$ was computed. Its zeros are shown in Fig. 1. The value of τ_{32} (cf. (2.6)) is 1.03626 to 5D. The coefficients γ_j in

$$\hat{Q}_{32}(z) p_6(z) := \sum_{j=0}^{38} \gamma_j z^j$$

satisfy $\gamma_j/\gamma_{j+1} = 1.03626$ for *all* $0 \leqq j \leqq 37$ except for $\gamma_{26}/\gamma_{27} = 0.944348$ and $\gamma_{32}/\gamma_{33} = 0.051895$. Two of the zeros of $\hat{Q}_{32}$ are roughly equal to the zeros of $z^2 + Rz + R^2$, where $R = 1.03626 = \tau_{32}$ (compare (4.7) in the proof of Theorem 1).

Note the circular pattern of the zeros of $\hat{Q}_{32}(z) \cdot p_6(z)$. This idea is used in the proof of Theorem 1, although the multiplier polynomials used there are not "optimal" at each stage. For example, using the technique of this proof on $p_6(z)$ above yields $\beta[Q_{48} p_6] = 2^{1/5} \approx 1.14870$, which is not as good as the result $\beta[\hat{Q}_{32} p_6] = 1.03626$ obtained from linear programming.

Example 2. $p_4(z) = (z^2 - \sqrt{3}z + 1)(z^2 + (\sqrt{2}/2)z + \frac{1}{4}) \in \hat{\pi}_4$.

For this polynomial (which is not in π_4^+), an optimum multiplier polynomial $\hat{Q}_{17}(z)$ was computed. Its zeros are shown in Fig. 2. Again, note the tendency of the optimal multiplier $\hat{Q}_{17}(z)$ to "fill out" the rings ($|z| = 1, |z| = \frac{1}{2}$) on which the zeros of the original polynomial lie.

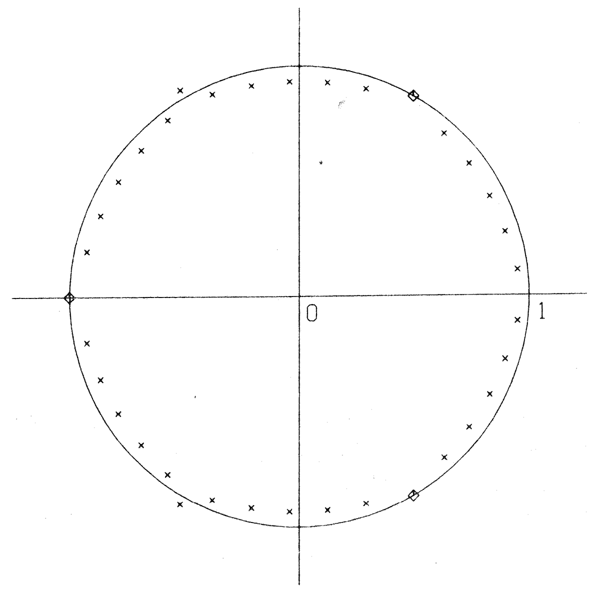

FIG. 1. *Zeros of* $p_6(z) = (z^3+1)^2$: *diamonds*; *zeros of optimal multiplier of degree* 32: *crosses* ($\tau_{32} \doteqdot 1.03626$).

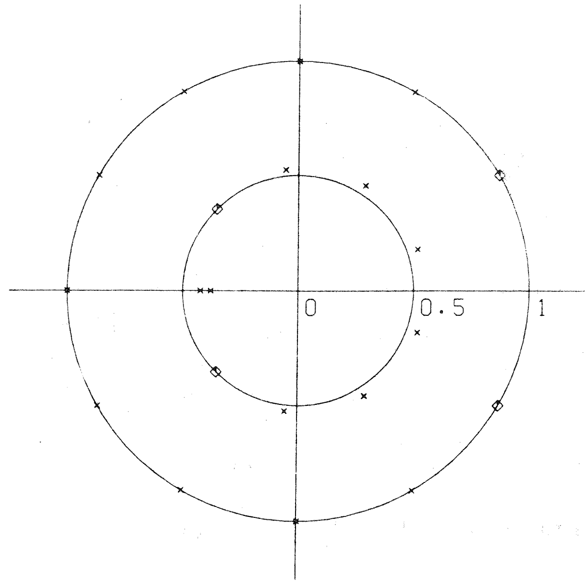

FIG. 2. *Zeros of* $p_4(z)=(z^2-\sqrt{3}\,z+1)(z^2+(\sqrt{2}/2)z+\frac{1}{4})$: *diamonds*; *zeros of optimal multiplier of degree* 17: *crosses* ($\tau_{17} \doteqdot 1.00034$).

3. Proofs of Proposition 1 and lemmas. We begin with the

Proof of Proposition 1. First, assume that $p_n(z)$ is any polynomial which has no zeros on the ray $[0, +\infty)$. Without loss of generality, we may assume that $p_n(z)$ is monic and moreover real, for if $(z-\zeta)$ is a factor of $p_n(z)$ with ζ not real, then both ζ and $\bar{\zeta}$ are not contained in $[0, +\infty)$ and we may consider $p_n(z)\cdot(z-\bar{\zeta})$ in place of $p_n(z)$ if $(z-\bar{\zeta})$ is not a factor of $p_n(z)$. Hence, we can express $p_n(z)$, by hypothesis, as

$$p_n(z)=\prod_{i=1}^{\alpha_1}(z+\delta_i)\prod_{j=1}^{\alpha_2}(z-r_je^{i\theta_j})(z-r_je^{-i\theta_j}),$$

or equivalently

$$p_n(z)=\prod_{i=1}^{\alpha_1}(z+\delta_i)\prod_{j=1}^{\alpha_2}(z^2-2r_j\cos\theta_jz+r_j^2), \tag{3.1}$$

where $\delta_i>0$ (if the first product is not vacuous), and where $r_j>0$ and $0<\theta_j<\pi$ (if the second product is not vacuous). If the second product is vacuous, then already $p_n \in \overset{+}{\pi}_n$. If the second product is not vacuous, consider the quadratic factor

$$z^2-2r_j\cos\theta_jz+r_j^2, \qquad r_j>0,\ 0<\theta_j<\pi.$$

If $\pi/2<\theta_j<\pi$, this quadratic factor is an element of $\overset{+}{\pi}_2$. If not, this quadratic factor divides

$$(z^2-2r_j\cos\theta_jz+r_j^2)(z^2+2r_j\cos\theta_jz+r_j^2)=z^4-2r_j^2\cos(2\theta_j)z^2+r_j^4.$$

If $\pi/4<\theta_j\leqq\pi/2$, this product, when multiplied by $(1+z)$, is then a polynomial in $\overset{+}{\pi}_5$. If $0<\theta_j\leqq\pi/4$, this process of doubling the argument θ_j can be continued, and eventually, since $\theta_j>0$, one obtains in this manner an element in some $\overset{+}{\pi}_\nu$. As this is true for each quadratic factor of (3.1), a polynomial multiplier can thus be found such that $Q_m(z)\cdot p_n(z)\in\overset{+}{\pi}_{m+n}$.

Conversely, supposing that $p_n(z)$ has a zero on the ray $[0, +\infty)$, the same is true for any product $Q_m(z)\cdot p_n(z)$, whence $Q_m(z)\cdot p_n(z)\notin \pi^{+}_{m+n}$ for any $Q_m(z)$. □

Before proceeding to the proof of Theorem 1 in § 4, we establish some results needed in the proof of that theorem.

LEMMA 1. *For any positive integer m, let $\{P_k(z)\}_{k=1}^m$ be any collection of m polynomials, each having positive coefficients and each being of degree at least unity. Then,*

$$\beta\left[\prod_{k=1}^{m} P_k\right] \leqq \sum_{k=1}^{m} \beta[P_k]. \tag{3.2}$$

Proof. The proof will be by induction on m. Obviously, (3.2) is valid for $m=1$. Assume, then, that (3.2) is true for m, and consider any $(m+1)$ polynomials $\{P_k(z)\}_{k=1}^{m+1}$, each having positive coefficients and degree at least unity. Calling

$$Q(z) := \prod_{k=1}^{m} P_k(z) = \sum_{i=0}^{\gamma} a_i z^i \quad \text{and} \quad P_{m+1}(z) := \sum_{j=0}^{\lambda} b_j z^j, \tag{3.3}$$

and, noting that rearrangements of the P_k's have no effect in (3.2), we may assume that $\gamma \geqq \lambda$. On setting

$$Q(z)\cdot P_{m+1}(z) := \sum_{k=0}^{\gamma+\lambda} c_k z^k,$$

then from (3.3) and the hypotheses of this lemma, we obtain

$$c_k = \sum_{j=0}^{k} a_j b_{k-j} > 0, \quad \text{for } k = 0, 1, \cdots, \gamma+\lambda,$$

where $a_j := 0$ for all $j > \gamma$ and where $b_j := 0$ for all $j > \lambda$. With the inductive hypothesis that $\beta[Q] \leqq \sum_{k=1}^{m} \beta[P_k]$, we must show that

$$c_k/c_{k+1} \leqq \beta[Q] + \beta[P_{m+1}] \quad \text{for all } k = 0, 1, \cdots, \gamma+\lambda-1. \tag{3.4}$$

This is done by considering the three cases: $0 \leqq k < \lambda$, $\lambda \leqq k < \gamma$, and $\gamma \leqq k \leqq \gamma+\lambda-1$. Since the proofs of the cases are similar, we consider for brevity only the case $\gamma \leqq k \leqq \gamma+\lambda-1$. In this case, we have

$$\begin{aligned}
c_k/c_{k+1} &= \left(\sum_{j=0}^{k} a_j b_{k-j}\right)\Big/\left(\sum_{j=0}^{k+1} a_j b_{k+1-j}\right) = \left(\sum_{j=k-\lambda}^{\gamma} a_j b_{k-j}\right)\Big/\left(\sum_{j=k+1-\lambda}^{\gamma} a_j b_{k+1-j}\right) \\
&= \left(a_{k-\lambda} b_\lambda + \sum_{j=k+1-\lambda}^{\gamma} a_j b_{k-j}\right)\Big/\left(\sum_{j=k+1-\lambda}^{\gamma} a_j b_{k+1-j}\right) \\
&\leqq \left(a_{k-\lambda} b_\lambda + \sum_{j=k+1-\lambda}^{\gamma} a_j \beta[P_{m+1}] b_{k+1-j}\right)\Big/\left(\sum_{j=k+1-\lambda}^{\gamma} a_j b_{k+1-j}\right) \\
&\leqq \frac{a_{k-\lambda} b_\lambda}{a_{k+1-\lambda} b_\lambda} + \beta[P_{m+1}] \leqq \beta[Q] + \beta[P_{m+1}] \leqq \sum_{k=1}^{m+1} \beta[P_k],
\end{aligned}$$

the last inequality making use of the inductive hypothesis $\beta[Q] \leqq \sum_{k=1}^{m} \beta[P_k]$, which completes the proof. □

Next, we establish a needed polynomial perturbation result.

LEMMA 2. *Given any complex number $re^{i\theta}$ with*

$$0 < \theta < \pi \quad \textit{and} \quad r > 0, \tag{3.5}$$

and given any sequence $\{s_\nu\}_{\nu=1}^{\infty}$ *of positive integers such that*

$$s_\nu \geqq 2 \quad \forall \nu \geqq 1, \quad \textit{and} \quad \lim_{\nu\to\infty} s_\nu = +\infty, \tag{3.6}$$

there exists a sequence of monic polynomials $\{\tilde{P}_{s_\nu}(z)\}_{\nu=1}^{\infty}$ *with* $\tilde{P}_{s_\nu} \in \pi_{s_\nu}^+$ *for all* ν *sufficiently large such that*

$$\begin{cases} \text{(i)} & (z-re^{i\theta})(z-re^{-i\theta}) \textit{ divides } \tilde{P}_{s_\nu}(z) \textit{ for all } \nu \geqq 1, \textit{ and} \\ \text{(ii)} & \lim_{\nu\to\infty} \beta[\tilde{P}_{s_\nu}] = r. \end{cases} \tag{3.7}$$

Proof. First, if we can show that $\{\tilde{P}_{s_\nu}(z)\}_{\nu=1}^{\infty}$ satisfies (3.7) for the special case $r=1$, then $\tilde{P}_{s_\nu}(z/r)$ will satisfy (3.7) for the general case. Thus, assuming $r=1$, we define the real monic polynomials $P_{s_\nu}(z)$ by

$$P_{s_\nu}(z) := 1+z+\cdots+z^{s_\nu} = (z^{s_\nu+1}-1)/(z-1), \tag{3.8}$$

whose simple zeros are $\zeta_k(s_\nu) := \exp[2\pi ki/(s_\nu+1)]$, $k=1, 2, \cdots, s_\nu$. For each ν sufficiently large, choose the distinct zero, say $\zeta_{k_1}(s_\nu)$, of $P_{s_\nu}(z)$ which best approximates $e^{i\theta}$ in the upper-half complex plane. It is geometrically clear that

$$|\zeta_{k_1}(s_\nu) - e^{i\theta}| \leqq \frac{\pi}{s_\nu+1}, \tag{3.9}$$

for all ν sufficiently large, say $\nu \geqq \nu_0$. With this choice, define

$$\tilde{P}_{s_\nu}(z) := P_{s_\nu}(z) \frac{(z-e^{i\theta})(z-e^{-i\theta})}{(z-\zeta_{k_1}(s_\nu))(z-\bar{\zeta}_{k_1}(s_\nu))} \quad \forall \nu \geqq \nu_0, \tag{3.10}$$

so that $\tilde{P}_{s_\nu}(z)$ is a monic polynomial in π_{s_ν}, and $(z-e^{i\theta})(z-e^{-i\theta})$ divides $\tilde{P}_{s_\nu}(z)$. In essence, the pair of simple zeros $\zeta_{k_1}(s_\nu)$ and $\bar{\zeta}_{k_1}(s_\nu)$ is "perturbed" to the two simple zeros $e^{\pm i\theta}$ of $\tilde{P}_{s_\nu}(z)$. Expressing $\tilde{P}_{s_\nu}(z)$ in the form

$$\tilde{P}_{s_\nu}(z) = \sum_{i=0}^{s_\nu} a_i(s_\nu;\theta) z^i \quad \forall \nu \geqq \nu_0, \tag{3.11}$$

and writing $a_i = a_i(s_\nu;\theta)$ and $\arg \zeta_{k_1}(s_\nu) =: \psi$, we obtain on cross-multiplying in (3.10) that

$$(z^2 - 2\cos\psi z + 1)\tilde{P}_{s_\nu}(z) = (z^2 - 2\cos\theta z + 1) P_{s_\nu}(z). \tag{3.12}$$

On equating coefficients of z^j on both sides, we have (cf. (3.11))

$$a_{j-2} - 2\cos\psi a_{j-1} + a_j = 2(1-\cos\theta), \qquad j = 1, 2, \cdots, s_\nu, \tag{3.13}$$

where $a_0 = 1 =: a_{-1}$. We write:

$$b_j := a_j - 1, \qquad j = -1, 0, \cdots, s_\nu; \tag{3.14}$$

then (3.13) becomes

$$b_{j-2} - 2\cos\psi b_{j-1} + b_j = 2(\cos\psi - \cos\theta), \qquad j = -1, 0, \cdots, s_\nu, \tag{3.15}$$

where $b_0 = b_{-1} = 0$. The solution of this linear difference equation can be verified to be

$$b_j = \frac{[\cos\psi - \cos\theta]\{\sin\psi + \sin j\psi - \sin(j+1)\psi\}}{(1-\cos\psi)\sin\psi}, \qquad -1 \leqq j \leqq s_\nu, \tag{3.16}$$

so that

$$(3.17)\qquad |b_j| \leqq \frac{3|\cos\psi - \cos\theta|}{(1-\cos\psi)\sin\psi}, \qquad -1 \leqq j \leqq s_\nu.$$

From our definition in (3.9) of $\zeta_{k_1}(s_\nu)$ and from (3.5), it then follows that there is a constant M, dependent only on θ, such that

$$(3.18)\qquad |b_j| \leqq M/s_\nu \quad \text{for all } -1 \leqq j \leqq s_\nu, \text{ and for all } s_\nu \geqq 2.$$

Recalling (3.14), we see that (3.18) implies that $|a_j - 1| \leqq M/s_\nu$, from which it follows that $\tilde{P}_{s_\nu}(z) \in \pi^+_{s_\nu}$ for all ν sufficiently large, as well as (cf. (3.7 (ii))) $\lim_{\nu\to\infty} \beta[\tilde{P}_{s_\nu}] = 1$. □

4. Proof of Theorem 1. Consider (cf. (1.11)) any $p_n(z) \in \hat{\pi}_n$ with $n \geqq 1$, and assume, without loss of generality, that $p_n(z)$ is monic, real, and is normalized so that (cf. (1.4))

$$\rho(p_n) = 1.$$

Writing

$$(4.1)\qquad p_n(z) = \prod_{i=1}^{n} (z - \zeta_i),$$

so that $|\zeta_i| \leqq 1$ for all i, we define

$$(4.2)\qquad P_t(w) := \prod_{i=1}^{n} (w - \zeta_i^t),$$

where t is any positive integer, and set

$$(4.3)\qquad S_i := \{t \in \mathbb{Z}_+ : \zeta_i^t \notin [0, +\infty)\}, \qquad 1 \leqq i \leqq n,$$

where $\mathbb{Z}_+$ denotes the set of all positive integers. Because $p_n(z) \in \hat{\pi}_n$, then $1 \in S_i$, and S_i is thus nonempty for all $1 \leqq i \leqq n$. Note that if some $\arg \zeta_i =: \theta_i$ is a rational multiple of π, i.e., (in lowest terms) $\theta_i = 2\pi\gamma/\delta$ where γ and δ are positive integers with $\gamma/\delta < 1$, then no multiple of δ is in S_i, while all $t \neq 0 \pmod{\delta}$ are in S_i. In this case, it is evident that

$$S_i = \mathbb{Z}_+ \backslash \{m\delta\}_{m=1}^{\infty}.$$

On the other hand, if some $\arg \zeta_i$ is not a rational multiple of π, then $S_i = \mathbb{Z}_+$. Consequently, since $p_n(z)$ is a fixed polynomial in $\hat{\pi}_n$, then

$$(4.4)\qquad \begin{cases} \text{(i)} & \bigcap_{i=1}^{n} S_i =: T = \{t_j\}_{j=1}^{\infty} \subset \mathbb{Z}_+, \quad \text{and} \\ \text{(ii)} & 1 = t_1 < t_2 < t_3 \cdots, \quad \text{with } \lim_{j\to\infty} t_j = +\infty. \end{cases}$$

We claim now that for each $t_j \in T$, there exists a polynomial $G_j(w)$ such that

$$(4.5)\qquad \begin{cases} \text{(i)} & G_j(w) \text{ is monic and has positive coefficients for all } j \geqq 1, \\ \text{(ii)} & P_{t_j}(w) \text{ of (4.2) divides } G_j(w) \text{ for all } j \geqq 1, \text{ and} \\ \text{(iii)} & \beta[G_j] \leqq n, \end{cases}$$

where n is the degree of $p_n(z)$ in (4.1). To see this, consider from (4.2) any factor $(w - \zeta_i^{t_j})$ of $P_{t_j}(w)$, where $t_j \in T$. If $\zeta_i^{t_j}$ is real, i.e., $\arg \zeta_i^{t_j} = \pi$, then this factor is just $(w + |\zeta_i^{t_j}|)$, since T can contain only odd integers in this case, and moreover,

$\beta[(w+|\zeta_i^{t_j}|)]=|\zeta_i^{t_j}|\leqq 1$. If $\zeta_i^{t_j}$ is not real, the reality of the polynomial $P_{t_j}(w)$ gives us that the product

$$(w-\zeta_i^{t_j})(w-(\bar{\zeta}_i)^{t_j}) \tag{4.6}$$

divides $P_{t_j}(w)$, where we may assume that $0<\arg\zeta_i^{t_j}<\pi$. Applying Lemma 2 to the product of (4.6) gives a polynomial $\tilde{P}_{i,j}(w)$ having (4.6) as a factor, such that $\tilde{P}_{i,j}(w)$ has positive coefficients, and such that $\beta[\tilde{P}_{i,j}]\leqq 2$. Thus, multiplying all these $\tilde{P}_{i,j}(w)$'s together, thereby forming $G_j(w)$, gives that $G_j(w)$ is monic with positive coefficients. Applying Lemma 1 to the product defining $G_j(w)$ gives $\beta[G_j(w)]\leqq n$, and by construction, $P_{t_j}(w)$ of (4.2) divides $G_j(w)$, thereby establishing (4.5).

Next, for each $R>0$, form the product

$$H_j(z;R):=\{R^{t_j-1}+R^{t_j-2}z+\cdots+z^{t_j-1}\}G_j(z^{t_j}), \tag{4.7}$$

for each $t_j\in T$, where $G_j(w)$ satisfies (4.5). Because $G_j(w)$ has all positive coefficients from (4.5 (i)), the polynomial $H_j(z;R)$ defined in (4.7) similarly has all positive coefficients, and the Eneström-Kakeya functional β of (1.2) can be directly applied to it. Note that the given $p_n(z)$ in $\hat{\pi}_n$ divides $H_j(z;R)$ (cf. (4.2) and (4.5 (ii)) for each choice of $R>0$. Now, it can be easily verified that

$$\beta[H_j(z;R)]=\max\{R;\Gamma_j/R^{t_j-1}\}, \tag{4.8}$$

where $\Gamma_j:=\beta[G_j]$. On equating R and Γ_j/R^{t_j-1}, i.e., on setting $R_j:=\Gamma_j^{1/t_j}$, we obtain from (4.8) that

$$\beta[H_j(z;R_j)]=\Gamma_j^{1/t_j}=(\beta[G_j])^{1/t_j},\quad \forall t_j\in T. \tag{4.9}$$

To complete the proof of Theorem 1, it thus remains from (4.9) to show that

$$\lim_{j\to\infty}(\beta[G_j])^{1/t_j}=1. \tag{4.10}$$

Since $p_n(z)$ divides $H_j(z;R_j)$ and since, by normalization, $\rho(p_n)=1$, then from (4.9) and (4.5 (iii)),

$$1=\rho(p_n)\leqq\beta[H_j(z;R_j)]=(\beta[G_j])^{1/t_j}\leqq(n)^{1/t_j},$$

which yields, by way of (4.4 (ii)), the desired result of (4.10). □

5. Proof of Theorem 2. First, assume that for $p_n(z)\in\hat{\pi}_n$ with $n\geqq 1$, there exists a polynomial $Q_m(z)$ in π_m with $Q_m(z)\cdot p_n(z)\in\overset{+}{\pi}_{m+n}$, such that $\beta[Q_mp_n]=\rho(p_n)$. Without loss of generality, we may, as in § 4, normalize to the case $\rho(p_n)=1$, i.e.,

$$\beta[Q_mp_n]=\rho(p_n)=1.$$

From (1.5), $\beta[Q_mp_n]\geqq\rho(Q_mp_n)\geqq\rho(p_n)=1$, so that

$$\beta[Q_mp_n]=\rho(Q_mp_n)=\rho(p_n)=1.$$

Hence, from Theorem B, all zeros of $Q_m(z)\cdot p_n(z)$ on $|z|=1$ are necessarily simple, which establishes the necessity of (1.14 (i)) of Theorem 2. Next, again from Theorem B, there is a positive integer $\bar{k}>1$ such that the zeros of $Q_m(z)\cdot p_n(z)$ on $|z|=1$ are precisely of the form

$$\exp\{2\pi ij/\bar{k}: j=1,2,\cdots,\bar{k}-1\}. \tag{5.1}$$

Evidently, each zero of $p_n(z)$ (as well as each zero of $Q_m(z)$) on $|z|=1$ is a (nonzero) rational multiple of 2π; i.e., if $\{\zeta_j\}_{j=1}^r$ denotes the set of all zeros of $p_n(z)$ on $|z|=1$, then

$$\arg \zeta_j = 2\pi n_j/d_j \quad \text{(in lowest terms)}, \tag{5.2}$$

where n_j and d_j are positive integers with $0<n_j<d_j$ for all $j=1, 2, \cdots, r$, thereby establishing the necessity of (1.14 (ii)) of Theorem 2.

Next, again from Theorem B, we have that, for some nonnegative integer l,

$$Q_m(z)\cdot p_n(z)=(1+z+z^2+\cdots+z^{\bar{k}-1})g_l(z^{\bar{k}}), \tag{5.3}$$

where $g_l(w)\in \pi_l^+$, and if $l\geqq 1$, all zeros of $g_l(w)$ lie in $|w|<1$, and $\beta[g_l]\leqq 1$. Clearly, the zeros of $p_n(z)$ on $|z|=1$ must be of the form (5.1), so that, for suitable integers ν_j,

$$\frac{n_j}{d_j}=\frac{\nu_j}{\bar{k}}, \qquad j=1, 2, \cdots, r.$$

Thus, if $D := \text{l.c.m.}\,\{d_j\}_{j=1}^r$, then D divides $\bar{k}$, whence $\bar{k}=\sigma D$ for some positive integer σ. Now, consider any zero ζ of $p_n(z)$ with $|\zeta|<1$. Evidently, $\zeta^{\bar{k}}$ is a zero of $g_l(w)$ from (5.3). But, since $g_l(w)\in\pi_l^+$, then $\zeta^{\bar{k}}=\zeta^{\sigma D}$ could not be contained in $[0, +\infty)$, which establishes the necessity of (1.14 (iii)).

Conversely, assume that $p_n\in\hat{\pi}_n$ with $n\geqq 1$, that $\rho(p_n)=1$, and that (1.14) is valid. Defining $\tilde{p}_{n-r}(z) := p_n(z)/\prod_{j=1}^r (z-\zeta_j)=\prod_{j=1}^{n-r}(z-\mu_j)$, where again $\{\zeta_j\}_{j=1}^r$ is the set of all zeros of $p_n(z)$ on $|z|=1$, then either $\tilde{p}_{n-r}(z)$ is a nonzero constant, or $\tilde{p}_{n-r}(z)$ is a polynomial of degree $n-r\geqq 1$, all of whose zeros lie in $|z|<1$. In the former case, hypothesis (1.14 (iii)) holds vacuously, while in the latter case, hypothesis (1.14 (iii)) implies that $\hat{p}_{n-r}(W) := \prod_{j=1}^{n-r}(W-\mu_j^{\bar{k}})$ is an element of $\hat{\pi}_{n-r}$. Now, applying Theorem 1 to $\hat{p}_{n-r}(W)\in\hat{\pi}_{n-r}$, shows that there exists a sequence of polynomials $\{Q_i(W)\}_{i=1}^\infty$ such that

$$\begin{cases} \text{(i)} & Q_i(W)\cdot\hat{p}_{n-r}(W) \text{ has positive coefficients for all } i\geqq 1, \text{ and} \\ \text{(ii)} & \lim_{i\to\infty}\beta[Q_i\cdot\hat{p}_{n-r}]=\rho(\hat{p}_{n-r}(W))<1. \end{cases} \tag{5.4}$$

To fix matters, choose from this sequence $\{Q_i(W)\}_{i=1}^\infty$ the polynomial $\tilde{Q}(W)$ of *least* degree such that

$$\begin{cases} \text{(i)} & \tilde{Q}(W)\cdot\hat{p}_{n-r}(W) \text{ has positive coefficients, and} \\ \text{(ii)} & \beta[\tilde{Q}\cdot\hat{p}_{n-r}]\leqq 1, \end{cases} \tag{5.5}$$

and set $g(W) := \tilde{Q}(W)\hat{p}_{n-r}(W)$.

Since $\prod_{j=1}^r(z-\zeta_j)$ divides, from (1.14 (ii)), the polynomial

$$1+z+z^2+\cdots+z^{\bar{k}-1}, \quad \text{where } \bar{k}=\sigma D,$$

it follows that there is a polynomial multiplier, $Q_m(z)$, such that

$$Q_m(z)\cdot p_n(z)=\{1+z+\cdots+z^{\bar{k}-1}\}g(z^{\bar{k}}), \tag{5.6}$$

where $g(W)$, from (5.5), has positive coefficients, $\beta[g]\leqq 1$, and all zeros of $g(W)$ lie in $|z|\leqq 1$. By this construction, one directly verifies that

$$\beta[Q_m p_n]=1=\rho(p_n), \tag{5.7}$$

which establishes (1.13). □

We remark that the construction in the proof of the sufficiency of conditions (1.14) of Theorem 2 gives the multiplier polynomial $Q_m(z)$ of *least* degree for satisfying (5.7).

To illustrate the result of Theorem 2, consider

Example 3. $p_3(z) := (1+z^2)(\frac{1}{2}+z)\in\hat{\pi}_3$.

The zeros of $p_3(z)$ are $\pm i$, $-\frac{1}{2}$, so that $\rho(p_3)=1$. For the zeros on $|z|=1$, their arguments are $2\pi/4$ and $6\pi/4$, whence (cf. (1.14 (iii))) $D=4$. But, since $(-\frac{1}{2})^{4\sigma}\in[0,+\infty)$ for *every* positive integer σ, hypothesis (1.14 (iii)) fails, and it is not possible to find a polynomial multiplier $Q_m(z)$ for which (1.13) is valid.

Example 4. $p_4(z)=(1+z^2)(\frac{1}{4}+\frac{1}{2}z+z^2)\in\hat{\pi}_4$.

The zeros of $p_4(z)$ are $\pm i$, and $\frac{1}{2}e^{\pm 2\pi i/3}$. As in the previous case, the zeros on $|z|=1$ have arguments $2\pi/4$ and $6\pi/4$, whence $D=4$, and as $(\frac{1}{2}e^{\pm 2\pi i/3})^{4\sigma}\notin[0,+\infty)$ for $\sigma=1, 2, 4, \cdots$, we choose $\sigma=1$. In this example, the conditions of (1.14) of Theorem 2 are valid, and with $Q_5(z):=(1+z)(\frac{1}{4}-\frac{1}{2}z+z^2)(\frac{1}{16}-\frac{1}{4}z+z^2)$, then

$$Q_5(z)p_4(z)=(1+z+z^2+z^3)g_2(z^4), \quad \text{with } g_2(w)=\frac{1}{256}+\frac{w}{16}+w^2,$$

so that

$$\beta[Q_5p_4]=1=\rho(p_4).$$

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