/29

ON INCOMPLETE POLYNOMIALS. II

E. B. SAFF AND R. S. VARGA

Reprinted from Pacific Journal of Mathematics Vol. 92, No. 1
1 9 8 1

PACIFIC JOURNAL OF MATHEMATICS
Vol. 92, No. 1, 1981

ON INCOMPLETE POLYNOMIALS. II

E. B. SAFF AND R. S. VARGA

The approximation of x^n by incomplete polynomials is studied, i.e., we consider the extremal problem

$$E_{n-k,k} = \inf\left\{\left\|x^n + \sum_{j=1}^{k} d_j x^{n-j}\right\|_{[0,1]} : (d_1, \cdots, d_k) \in \boldsymbol{R}^k\right\}, \quad n \geqq k,$$

for the supremum norm on $[0, 1]$. **We show that, for** k **fixed,** $n^k E_{n-k,k} \to \varepsilon_k$ **as** $n \to \infty$, **where**

$$\varepsilon_k = \inf\left\{\left\| e^{-t}\left(t^k + \sum_{j=0}^{k-1} a_j t^j\right)\right\|_{[0,+\infty)} : (a_0, \cdots, a_{k-1}) \in \boldsymbol{R}^k\right\}.$$

A generalization of this result for the case of lacunary polynomial approximation is given, as well as inequalities for $E_{n-k,k}$ **and** ε_k. **Furthermore, we prove that for any polynomial** $P(t)$ **of degree at most** k, **there holds for the supremum norm** $\|e^{-t}P(t)\|_{[0,+\infty)} = \|e^{-t}P(t)\|_{[0,2k]}$.

1. Introduction. In this note, we continue our investigation [6], [7], [8], [3], of incomplete polynomials, a subject first introduced by G. G. Lorentz [4]. Following the notation of [7], if π_n denotes the set of all real polynomials of degree at most n, then for each pair (s, k) of nonnegative integers, $\pi_{s,k}$ denotes the set of polynomials

$$\pi_{s,k} := \{x^s q_k(x) : q_k \in \pi_k\}, \tag{1.1}$$

so that $\pi_{s,k} \subset \pi_{s+k}$. A polynomial in $\pi_{s,k}$ is called an *incomplete polynomial* of type (s, k). For any set $K \subset \boldsymbol{R}$, the norm $\|\cdot\|_K$ shall denote the supremum norm on K, i.e., $\|g\|_K := \sup\{|g(x)| : x \in K\}$. Setting

$$E_{s,k} := \inf\{\|x^s(x^k - q(x))\|_{[0,1]} : q \in \pi_{k-1}\}, \quad \pi_{-1} := \{0\}, \tag{1.2}$$

it is known [7] that, for each pair (s, k), there exists a unique monic polynomial $Q_{s,k}(x) \in \pi_{s,k}$ of exact degree $s + k$, such that $\|Q_{s,k}\|_{[0,1]} = E_{s,k}$.

In a recent paper, Borosh, Chui, and Smith [1] established that for any positive integer k, there exist positive constants $\sigma_1(k)$ and $\sigma_2(k)$ such that

$$\sigma_1(k) \leqq n^k E_{n-k,k} \leqq \sigma_2(k), \quad \forall n > k. \tag{1.3}$$

They also proved that the coefficients of the extremal polynomials $Q_{n-k,k}(x)$ are bounded as $n \to \infty$.

One aim of this note is to derive (cf. (3.3)) explicit upper and lower bounds for $n^k E_{n-k,k}$ for all $n > k$. But more importantly, we prove (in Corollary 2.3) that the limit of $n^k E_{n-k,k}$ as $n \to \infty$ exists and is given precisely by

$$(1.4) \qquad \lim_{n\to\infty} n^k E_{n-k,k} = \varepsilon_k , \quad \forall k \geqq 0 ,$$

where

$$(1.5) \qquad \varepsilon_k := \inf\{\|e^{-t}(t^k - p(t))\|_{[0,+\infty)} : p \in \pi_{k-1}\} .$$

Since, after suitable normalization, the extremal polynomials for (1.2) are L^∞-analogs of Jacobi polynomials, and the extremal polynomials for (1.5) are L^∞-versions of Laguerre polynomials, our essential contribution is to show, as in the L^2-setting, that L^∞-Laguerre polynomials can be obtained as the limit of certain L^∞-Jacobi polynomials.

A generalization of the inequalities (1.3) to the case of lacunary polynomials is also proved in [1]. As a consequence of our main theorem, we derive the best possible result for this case as well (cf. Corollary 2.2).

The outline of the paper is as follows. In §2, we prove our main result and deduce as corollaries the limit (1.4) and its analog for lacunary polynomials. In §3, we derive inequalities for incomplete polynomials, and in §4 we study the quantities ε_k of (1.5). We also prove in §4 (cf. Theorem 4.2) that for *any* polynomial $P \in \pi_k$, there holds

$$(1.6) \qquad \|e^{-t}P(t)\|_{[0,+\infty)} = \|e^{-t}P(t)\|_{[0,2k]} .$$

Moreover, the interval $[0, 2k]$ is best possible in an asymptotic sense, as $k \to \infty$.

2. The main result and its consequences.

Our primary objective is to prove

THEOREM 2.1. *Let the $k + 1$ integers $0 \leqq \mu_1 < \mu_2 < \cdots < \mu_k < m$ be fixed, and, for each nonnegative integer n, set*

$$(2.1) \qquad E_n := \inf\left\{\left\|x^n\left(x^m - \sum_{j=1}^{k} c_j x^{\mu_j}\right)\right\|_{[0,1]} : (c_1, \cdots, c_k) \in \boldsymbol{R}^k\right\} .$$

Then,

$$(2.2) \qquad \lim_{n\to\infty} n^k E_n = \frac{\varepsilon_k}{k!} \prod_{j=1}^{k} (m - \mu_j) ,$$

where ε_k is defined in (1.5).

Before proceeding with the proof, we remark that by Descartes' rule of signs, the linear space generated by the functions $x^{n+\mu_1}$, $x^{n+\mu_2}$, $\cdots$, $x^{n+\mu_k}$ satisfies the Haar condition on $(0, 1]$, and hence (cf. [5, p. 16]), the extremal problem (2.1) has a unique solution for each $n \geqq 0$, i.e., there exist unique constants $c_j^*(n)$, $j = 1, \cdots, k$, such that

$$E_n = \left\| x^n\left(x^m - \sum_{j=1}^{k} c_j^*(n)x^{\mu_j}\right)\right\|_{[0,1]}, \quad \forall n \geqq 0 . \tag{2.3}$$

For notational convenience, we set

$$p_n^*(x) := x^m - \sum_{j=1}^{k} c_j^*(n)x^{\mu_j}, \quad \forall n \geqq 0 . \tag{2.4}$$

The proof of Theorem 2.1 requires the following lemmas:

LEMMA 1. (*Lorentz* [4], *Saff and Varga* [6]). *Let* $P(x)(\not\equiv 0)$ *be an incomplete polynomial of type* (s, m), *where* $s + m > 0$. *If* $|P(\zeta)| = \|P\|_{[0,1]}$ *with* $\zeta \in [0, 1]$, *then* $s^2/(s + m)^2 \leqq \zeta \leqq 1$.

LEMMA 2. *For each* n, *the polynomial* $x^n p_n^*(x)$ (cf. (2.4)) *has an alternation set of* $k + 1$ *points in* $[n^2/(n + m)^2, 1]$, i.e., *there exist points* $\{\zeta_{i,n}\}_{i=0}^{k}$ *with*

$$n^2/(n + m)^2 \leqq \zeta_{0,n} < \cdots < \zeta_{k,n} \leqq 1 \tag{2.5}$$

such that

$$\zeta_{i,n}^n p_n^*(\zeta_{i,n}) = (-1)^{k-i}E_n, \quad i = 0, 1, \cdots, k . \tag{2.6}$$

Consequently, $p_n^*(x)$ *has* k *distinct zeros in* $(n^2/(n + m)^2, 1)$, *and, as* $n \to \infty$, *these* k *zeros all tend to* $x = 1$.

Proof. From the fundamental property of Haar system approximation (cf. [5, p. 20]), $x^n p_n^*(x)$ has an alternation set consisting of $k + 1$ points in $[0, 1]$ and, by Lemma 1, these points must all belong to $[n^2/(n + m)^2, 1]$. □

LEMMA 3. *Set* $\mu_{k+1} := m$. *Then, for the polynomials* $p_n^*(x)$ *of* (2.4), *we have, uniformly on each compact set of* $\boldsymbol{R}$,

$$\lim_{n\to\infty} p_n^*(x) = \frac{1}{A}\det\begin{bmatrix} C(\mu_1, 0) & C(\mu_2, 0) & \cdots & C(\mu_{k+1}, 0) \\ C(\mu_1, 1) & C(\mu_2, 1) & \cdots & C(\mu_{k+1}, 1) \\ \vdots & \vdots & & \vdots \\ C(\mu_1, k-1) & C(\mu_2, k-1) & \cdots & C(\mu_{k+1}, k-1) \\ x^{\mu_1} & x^{\mu_2} & \cdots & x^{\mu_{k+1}} \end{bmatrix}, \tag{2.7}$$

where $C(\mu_j, i)$ *denotes the binomial coefficient* $\binom{\mu_j}{i}$, *and where* $A := \det[a_{i,j}]$, $a_{i,j} := C(\mu_j, i-1)$, $i, j = 1, 2, \cdots, k$.

Here and below, we adopt the usual convention that $C(\mu_j, i) = 0$ if $\mu_j < i$.

Proof. Let $x_1^{(n)}, x_2^{(n)}, \cdots, x_k^{(n)}$ denote the k zeros of $p_n^*(x)$ in $(0, 1)$ (cf. Lemma 2). Then, we claim that $p_n^*(x)$ can be expressed as

$$(2.8)\qquad p_n^*(x) = \det\begin{bmatrix} a_{1,1}^{(n)} & a_{1,2}^{(n)} & \cdots & a_{1,k+1}^{(n)} \\ a_{2,1}^{(n)} & a_{2,2}^{(n)} & \cdots & a_{2,k+1}^{(n)} \\ \vdots & \vdots & & \vdots \\ a_{k,1}^{(n)} & a_{k,2}^{(n)} & \cdots & a_{k,k+1}^{(n)} \\ x^{\mu_1} & x^{\mu_2} & \cdots & x^{\mu_{k+1}} \end{bmatrix} \div \det\begin{bmatrix} a_{1,1}^{(n)} & a_{1,2}^{(n)} & \cdots & a_{1,k}^{(n)} \\ a_{2,1}^{(n)} & a_{2,2}^{(n)} & \cdots & a_{2,k}^{(n)} \\ \vdots & \vdots & & \vdots \\ a_{k,1}^{(n)} & a_{k,2}^{(n)} & \cdots & a_{k,k}^{(n)} \end{bmatrix},$$

where each entry $a_{i,j}^{(n)}$ is the $(i-1)$-st order divided difference of the function $f_j(x) := x^{\mu_j}$ in the points $x_1^{(n)}, x_2^{(n)}, \cdots, x_k^{(n)}$, i.e.,

$$(2.9)\qquad a_{i,j}^{(n)} := f_j[x_1^{(n)}, x_2^{(n)}, \cdots, x_i^{(n)}],\ i = 1, \cdots, k,\ j = 1, \cdots, k+1 .$$

To see this, the polynomial defined by the right side of (2.8) is evidently monic, vanishing for $x_1^{(n)}, \cdots, x_k^{(n)}$, and is thus $p_n^*(x)$ by uniqueness. Now as $n \to \infty$, we have from Lemma 2 that $x_i^{(n)} \to 1$ for each $i = 1, \cdots, k$, whence

$$(2.10)\qquad \lim_{n\to\infty} a_{i,j}^{(n)} = \frac{f_j^{(i-1)}(1)}{(i-1)!} = C(\mu_j, i-1),\ i = 1, \cdots, k,\ j = 1, \cdots, k+1 .$$

Furthermore, the limit of the denominator determinants in (2.8), which is A from (2.10), is different from zero because it can be expressed as a nonzero constant times the Vandermonde determinant in the distinct points $\mu_1, \mu_2, \cdots, \mu_k$(cf. [2, p. 47]). Thus, (2.7) follows from (2.10) and (2.8). □

We note from (2.7) that $\lim_{n\to\infty} p_n^{*(k)}(1) = k!\,B/A$, where $B := \det[b_{i,j}]$ with $b_{i,j} = C(\mu_j, i-1)$, $i, j = 1, 2, \cdots, k+1$. Since B equals $\prod_{i=0}^{k}(i!)^{-1}$ times the Vandermonde determinant in the points $\mu_1, \mu_2, \cdots, \mu_{k+1} = m$, and since A equals $\prod_{i=0}^{k-1}(i!)^{-1}$ times the Vandermonde determinant in the points $\mu_1, \mu_2, \cdots, \mu_k$, we have that

$$(2.11)\qquad \lim_{n\to\infty} p_n^{*(k)}(1) = \prod_{j=1}^{k}(m - \mu_j) .$$

This fact will be useful in the

Proof of Theorem 2.1. As stated in Lemma 2, the polynomial $x^n p_n^*(x)$ has $k+1$ alternations in $[n^2/(n+m)^2, 1]$. Hence on replacing x by $1-t/n$, the polynomial $(1-t/n)^n q_n(t)$, where $q_n(t) := p_n^*(1-t/n)$, has $k+1$ alternations in the interval

$$0 \leqq t \leqq n\left(1-\frac{n^2}{(n+m)^2}\right) < 2m . \tag{2.12}$$

Furthermore, for each fixed $\rho \geqq 2m$, we have (cf. (2.3))

$$n^k E_n = n^k \| x^n p_n^*(x) \|_{[0,1]} = \left\| \left(1-\frac{t}{n}\right)^n n^k q_n(t) \right\|_{[0,\rho]} , \quad \forall n \geqq e . \tag{2.13}$$

Next, we claim that the sequence $\{n^k q_n(t)\}_{n=1}^\infty$ is uniformly bounded on compact sets of the real line. Indeed, by Lemma 2, we can write

$$p_n^*(x) = P_n(x) \prod_{i=1}^k (x - x_i^{(n)}) , \tag{2.14}$$

where $P_n(x)$ is a monic polynomial of degree $m-k$, and $n^2/(n+m)^2 < x_i^{(n)} < 1$ for each $i = 1, \cdots, k$. Thus,

$$n^k q_n(t) = (-1)^k P_n\left(1-\frac{t}{n}\right) \prod_{i=1}^k \left(t - t_i^{(n)}\right) , \quad t_i^{(n)} := n(1 - x_i^{(n)}) . \tag{2.15}$$

Next, on differentiating k-times the product in (2.14) via Leibniz's formula, we obtain

$$\lim_{n\to\infty} P_n\left(1-\frac{t}{n}\right) = \frac{1}{k!} \lim_{n\to\infty} p_n^{*(k)}\left(1-\frac{t}{n}\right) = \frac{1}{k!} \lim_{n\to\infty} p_n^{*(k)}(1) ,$$

uniformly on each compact subset of $\boldsymbol{R}$, the last equality following from (2.7). Thus, from (2.11),

$$\lim_{n\to\infty} P_n\left(1-\frac{t}{n}\right) = \frac{1}{k!} \prod_{j=1}^k (m - \mu_j) =: \gamma , \tag{2.16}$$

so that, with (2.15), and the fact that each $t_i^{(n)} \in (0, 2m)$ it follows, as claimed, that $\{n^k q_n(t)\}_{n=1}^\infty$ is uniformly bounded on compact sets of the real line.

Now, let $q^*(t)$ be any limit function of the sequence $\{n^k q_n(t)\}_{n=1}^\infty$. Then by (2.15) and (2.16), $q^*(t)$ is a polynomial of degree k of the form $q^*(t) = (-1)^k \gamma t^k + \cdots$. Since $e^{-t} q^*(t) (\not\equiv 0)$ is a limit function on $[0, \rho]$ of the sequence $\{(1-t/n)^n n^k q_n(t)\}_{n=1}^\infty$, and since each function in this sequence has $k+1$ alternation points in $[0, \rho]$, the same must be true of $e^{-t} q^*(t)$. Consequently, $q^*(t)$ must be the unique solution of the extremal problem

$$\varepsilon(\rho) := \inf \{ \| e^{-t} q(t) \|_{[0,\rho]} : q(t) = (-1)^k \gamma t^k + \cdots \in \pi_k \} . \tag{2.17}$$

But this implies that $\{n^k q_n(t)\}_{n=1}^{\infty}$ has a unique limit function and, from (2.13), that

$$\lim_{n\to\infty} n^k E_n = \|e^{-t}q^*(t)\|_{[0,\rho]} = \varepsilon(\rho) . \tag{2.18}$$

Finally, since (2.18) holds for all $\rho \geqq 2m$, we have with (1.5) and (2.16) that

$$\begin{aligned}\varepsilon(\rho) &= \inf\{\|e^{-t}q(t)\|_{[0,+\infty)}: q(t) = (-1)^k\gamma t^k + \cdots \in \pi_k\}\\ &= \gamma\varepsilon_k = \frac{\varepsilon_k}{k!}\prod_{j=1}^{k}(m-\mu_j) .\end{aligned}$$

This, coupled with (2.18), gives the desired result of (2.2). $\square$

As applications of Theorem 2.1, we now give the sharp improvements of the results in [1]. For this purpose, let $k > 0$ be a fixed integer, and let $\underline{\lambda}_n := (\lambda_1(n), \lambda_2(n), \cdots, \lambda_k(n))$, where the integers $\lambda_j(n)$ satisfy

$$0 \leqq \lambda_1(n) < \lambda_2(n) < \cdots < \lambda_k(n) < n .$$

Then, for the extremal problem

$$\mathscr{E}(\underline{\lambda}_n) := \inf\left\{\left\|x^n + \sum_{j=1}^{k} b_j x^{\lambda_j(n)}\right\|_{[0,1]} : (b_1, \cdots, b_k) \in \boldsymbol{R}^k\right\} , \tag{2.19}$$

we have

COROLLARY 2.2. *If $n - \lambda_1(n) \leqq D$ and $C \leqq n - \lambda_k(n)$ for all large n, where C, D are integers and $C \geqq 1$, then*

$$\binom{C+k-1}{k}\varepsilon_k \leqq \varliminf_{n\to\infty} n^k\mathscr{E}(\underline{\lambda}_n) \leqq \varlimsup_{n\to\infty} n^k\mathscr{E}(\underline{\lambda}_n) \leqq \binom{D}{k}\varepsilon_k , \tag{2.20}$$

where ε_k is defined in (1.5). Moreover, these inequalities are best possible.

Proof. Since

$$\mathscr{E}(\underline{\lambda}_n) = \inf\left\{\left\|x^{n-D}\left(x^D + \sum_{j=1}^{k} b_j x^{\mu_j}\right)\right\|_{[0,1]} : (b_1, \cdots, b_k) \in \boldsymbol{R}^k\right\} ,$$

where $\mu_j = \mu_j(n) := \lambda_j(n) - n + D$, then $\mathscr{E}(\underline{\lambda}_n)$ is of the form (2.1) with $m = D$ and n replaced by $n - D$. From the hypotheses on the $\lambda_j(n)$, we note that $0 \leqq \mu_1 < \mu_2 < \cdots < \mu_k \leqq D - C$ for all n large. Hence, by Theorem 2.1, the set $\mathscr{L}$ of possible limit points for the sequence $n^k\mathscr{E}(\underline{\lambda}_n)$ is given by

$$\mathscr{L} := \left\{\frac{\varepsilon_k}{k!} \prod_{j=1}^{k} (D - \mu_j) : 0 \leqq \mu_1 < \mu_2 < \cdots < \mu_k \leqq D - C\right\}.$$

As the largest possible limiting value occurs when $\mu_j = j - 1$, $j = 1, \cdots, k$, and the smallest limit occurs when $\mu_j = D - C + j - k$, $j = 1, \cdots, k$, the inequalities (2.20) follow. □

We remark that, under the assumptions of Corollary 2.2, if $b_j^* = b_j^*(n)$ denotes the extremal coefficients for (2.19), then Lemma 3 implies that these coefficients are bounded as $n \to \infty$. The precise statements of these bounds are left for the reader.

For the case when $D = k$ and $C = 1$, Corollary 2.2 immediately gives

COROLLARY 2.3. *For each fixed positive integer k, the quantities*

$$E_{n-k,k} = \inf \left\{\left\|x^n + \sum_{j=1}^{k} d_j x^{n-j}\right\|_{[0,1]} : (d_1, \cdots, d_k) \in R^k\right\}, \quad n \geqq k, \tag{2.21}$$

satisfy

$$\lim_{n\to\infty} n^k E_{n-k,k} = \varepsilon_k, \tag{2.22}$$

where ε_k is defined in (1.5).

REMARK 1. If $d_j^* = d_j^*(n)$, $j = 1, \cdots, k$, denotes the extremal coefficients for (2.21), then since the polynomials $x^k + \sum_{j=1}^{k} d_j^* x^{k-j}$ approach $(x-1)^k$ as $n \to \infty$, we have

$$\lim_{n\to\infty} d_j^* = (-1)^j \binom{k}{j}, \quad j = 1, \cdots, k. \tag{2.23}$$

REMARK 2. By using a slightly different method of proof, it can also be shown that

$$(n-k)^k E_{n-k,k} \leqq \varepsilon_k, \text{ for all } n \geqq k. \tag{2.24}$$

3. Inequalities for incomplete polynomials. We now obtain estimates for the quantities $E_{s,k}$ defined in (1.2).

THEOREM 3.1. *For any pair (s, k) of integers with $s \geqq 1$, $k \geqq 0$, we have*

$$\binom{k-1/2}{k}\binom{2s+2k-1}{k}^{-1} \leqq E_{s,k} \leqq \binom{2s+2k}{k}^{-1}. \tag{3.1}$$

Proof. The upper bound for $E_{s,k}$ in (3.1) follows directly from [7, eq. (30)]. To establish the lower bound in (3.1), we consider the polynomials

$$V_{s,k}(x) := \binom{2s+2k-1}{k}^{-1} x^s P_k^{(-1/2,\,2s-1/2)}(2x-1)\ ,$$

where $P_k^{(\alpha,\beta)}(t)$ denotes, as usual, the Jacobi polynomial. From Szegö [9, p. 63], $V_{s,k}(x)$ is monic of exact degree $s+k$. It is proved in Lemmas 3.2, 3.3 of [6] that $V_{s,k}(x)$ attains positive and negative relative extrema on $(0, 1]$ alternately at $k+1$ points $\hat{\xi}_i$, with $0 < \hat{\xi}_0 < \hat{\xi}_1 < \cdots < \hat{\xi}_k = 1$, and furthermore that $\min_{0 \leqq i \leqq k} |V_{s,k}(\hat{\xi}_i)| = |V_{s,k}(1)|$. Thus, on regarding (1.2) as weighted best approximation from π_{k-1} on $(0, 1]$, the theorem of de la Vallée Poussin [5, p. 82] implies that $E_{s,k} \geqq |V_{s,k}(1)|$. Again from Szegö [9, p. 58], $V_{s,k}(1)$ can be directly determined, so that

$$E_{s,k} \geqq |V_{s,k}(1)| = \binom{2s+2k-1}{k}^{-1}\binom{k-1/2}{k}\ . \qquad \square$$

Since $\binom{k-1/2}{k} = (2k)!/2^{2k}(k!)^2$, we obtain, on replacing s by $n-k$ in (3.1), that

$$\left(\frac{2n}{2n-k}\right)\frac{(2k)!}{2^{2k}(k!)^2} \leqq \binom{2n}{k} E_{n-k,k} \leqq 1\ , \quad \forall n > k\ . \tag{3.2}$$

In particular, (3.2) yields

$$\frac{(2k)!}{k!\,2^{3k}} \leqq n^k E_{n-k,k} \leqq (k+1)^k \binom{2k+2}{k}^{-1}\ , \quad \forall n > k\ , \tag{3.3}$$

which gives explicit estimates for the positive constants $\sigma_1(k)$, $\sigma_2(k)$ in (1.3).

Concerning bounds for the coefficients of arbitrary incomplete polynomials, we mention a simple consequence of Proposition 4 in [7]. For this purpose, we define

$$T_{s,k}(x) := Q_{s,k}(x)/E_{s,k} = x^s \sum_{i=0}^{k} \alpha_i^{(s,k)} x^i\ , \tag{3.4}$$

where, as stated in §1, $Q_{s,k}(x) \in \pi_{s,k}$ is the unique extremal polynomial for (1.2). Furthermore, let $\xi_0^{(s,k)} < \xi_1^{(s,k)} < \cdots < \xi_k^{(s,k)} = 1$ denote the alternation set for $Q_{s,k}(x)$ in $[0, 1]$ (cf. [7, Proposition 2]). Then, we have

THEOREM 3.2. *If $p(x) = x^s \sum_{i=0}^{k!} a_i x^i \in \pi_{s,k}$ and if $M \geqq \max\{|p(\xi_i^{(s,k)})|: 0 \leqq i \leqq k\}$, then*

$$|a_i| \leqq M|\alpha_i^{(s,k)}| , \quad \forall\ 0 \leqq i \leqq k , \tag{3.5}$$

where the $\alpha_i^{(s,k)}$ *are the coefficients in* (3.4).

Proof. In [7, Proposition 4], it is shown that the above hypotheses imply

$$\begin{cases} |p(x)| \leqq M|T_{s,k}(x)|, \forall x \notin (\xi_0^{(s,m)}, 1), \text{ and} \\ |p^{(\nu)}(x)| \leqq M|T_{s,k}^{(\nu)}(x)|, \forall x \notin (0, 1), \nu = 1, 2, \cdots . \end{cases} \tag{3.6}$$

Since $a_i = p^{(s+i)}(0)/(s+i)!$, and since (3.6) in particular holds when $x = 0$, the inequalities (3.5) follow. □

4. Inequalities for polynomials on $[0, +\infty)$. As a useful analog of the first inequality of (3.6) for the interval $[0, +\infty)$, let $q_k^*(t) = t^k + \cdots$, be the unique polynomial in π_k such that (cf. (1.5))

$$\varepsilon_k = ||e^{-t}q_k^*(t)||_{[0,+\infty)} ,$$

and let

$$0 = \eta_0^{(k)} < \eta_1^{(k)} < \cdots < \eta_k^{(k)} \tag{4.1}$$

be the unique $k+1$ alternation points for $e^{-t}q_k^*(t)$ in $[0, +\infty)$, i.e., $e^{-t}|q_k^*(t)| = \varepsilon_k$ for $t \in [0, +\infty)$ only if $t = \eta_i^{(k)}$ for some $0 \leqq i \leqq k$. Then, by applying the same reasoning as in the proof of [7, Prop. 4], we have

THEOREM 4.1. *If* $p \in \pi_k$ *and if* $M \geqq \max\{|e^{-\eta_i^{(k)}}p(\eta_i^{(k)})|: 0 \leqq i \leqq k\}$, *then*

$$|p(t)| \leqq M|q_k^*(t)|/\varepsilon_k , \qquad \forall t \notin (0, \eta_k^{(k)}) . \tag{4.2}$$

As a consequence of Theorem 4.1 and the fact (cf. (2.12) with $m = k$) that $\eta_k^{(k)} \leqq 2k$, we immediately have (cf. [7, Corollary 5]) the first part of

THEOREM 4.2. *If* $p \in \pi_k$ *with* $p \not\equiv 0$, *and if* $|e^{-\zeta}p(\zeta)| = ||e^{-t}p(t)||_{[0,+\infty)}$, *then* $\zeta \leqq \eta_k^{(k)} \leqq 2k$; *in particular,*

$$||e^{-t}p(t)||_{[0,+\infty)} = ||e^{-t}p(t)||_{[0,2k]} = ||e^{-t}p(t)||_{[0,\eta_k^{(k)}]} . \tag{4.3}$$

Moreover, the quantity $2k$ *is asymptotically best possible as* $k \to \infty$, *in the sense that there exists a constant* $\mu = 2.945\,820\cdots$ *for which*

$$2k - \mu \cdot k^{1/3} + o(k^{1/3}) \leqq \eta_k^{(k)} \leqq 2k , \qquad \text{as } k \to \infty . \tag{4.4}$$

Proof. To establish the result of (4.4), consider the Laguerre polynomial $L_k^{(-1/2)}(x)$ for each $k \geqq 1$. It is known (cf. Szegö [9, p. 176]) that $|L_k^{(-1/2)}(0)|$ and the successive relative maxima of $e^{-t}|L_k^{(-1/2)}(2t)|$ for $t \geqq 0$ form an increasing sequence for each k. Calling the abscissa of the last relative maximum $\hat{\zeta}_k$, then

$$e^{-\hat{\zeta}_k}|L_k^{(-1/2)}(2\hat{\zeta}_k)| = \|e^{-t}L_k^{(-1/2)}(2t)\|_{[0,+\infty)}, \tag{4.5}$$

which, from Theorem 4.2, implies that $\hat{\zeta}_k \leqq \eta_k^{(k)}$. Now, it is evident that $2\hat{\zeta}_k$ exceeds x_k, the largest zero of $L_k^{(-1/2)}(x)$, where x_k is known (cf. [9, p. 132]) to satisfy

$$x_k = 4k - \omega k^{1/3} + o(k^{1/3}), \qquad \text{as } k \to \infty, \tag{4.6}$$

with $\omega = 5.891\,639 \cdots$. But then, the inequalities $x_k/2 < \zeta_k \leqq \eta_k^{(k)} \leqq 2k$, coupled with (4.6), give (4.4). $\square$

For reference purposes, the numerical values of $\eta_k^{(k)}$ of (4.1) are given in Table I below for $1 \leqq k \leqq 11$.

We now consider estimates for the quantities ε_k. Since $n^k E_{n-k,k} \to \varepsilon_k$ as $n \to \infty$, we deduce from the inequalities (3.2) the result of

THEOREM 4.3. *For each integer* $k \geqq 0$,

$$\frac{(2k)!}{2^{2k}(k!)^2} \leqq \frac{2^k}{k!}\varepsilon_k \leqq 1. \tag{4.7}$$

For k large, this implies, via Stirling's formula, that

$$\frac{1}{\sqrt{\pi}} \leqq \liminf_{k\to\infty} \sqrt{k}\,\frac{2^k}{k!}\varepsilon_k, \quad \text{while} \quad \limsup_{k\to\infty} \frac{2^k}{k!}\varepsilon_k \leqq 1. \tag{4.8}$$

However, the upper bound in (4.8) can easily be improved. For this purpose, let $L_k^{(\alpha)}(t)$ denote, as usual, the Laguerre polynomial in π_k with respect to the weight function $e^{-t}t^\alpha$ on $[0, +\infty)$ (cf. [9, p. 100]), where $\alpha > -1$. Since $L_k^{(\alpha)}(t) = (-1)^k t^k/k! + \cdots \in \pi_k$, and since

$$\|e^{-t/2}L_k^{(\alpha)}(t)\|_{[0,+\infty)} = \|e^{-t}L_k^{(\alpha)}(2t)\|_{[0,+\infty)} = \left\|e^{-t}\left\{\frac{(-1)^k 2^k}{k!}t^k + \cdots\right\}\right\|_{[0,+\infty)}$$

we have, by definition of ε_k, that

$$\frac{2^k}{k!}\varepsilon_k \leqq \|e^{-t/2}L_k^{(\alpha)}(t)\|_{[0,+\infty)}, \ \forall \alpha > -1. \tag{4.9}$$

Now, in the case when $\alpha = -1/2$, it is known [9, p. 240] that asymptotically

$$\|e^{-t/2}L_k^{(-1/2)}(t)\|_{[0,+\infty)} \sim \frac{(18)^{1/6}\tau}{\pi k^{1/3}}, \qquad \text{as } k \to \infty, \tag{4.10}$$

where τ is the maximum of the Airy's function $A(t)$ for $t \in \boldsymbol{R}$. Thus, from (4.9), we have

$$\limsup_{k\to\infty} k^{1/3}\frac{2^k}{k!}\varepsilon_k \leqq (18)^{1/6}\tau/\pi = 0.6017\cdots . \tag{4.11}$$

In (4.8) and (4.11), lower and upper estimates for $2^k\varepsilon_k/k!$ are given which do not have the same asymptotic behavior as $k \to \infty$. It is an open question if there exist constants α with $1/3 \leqq \alpha \leqq 1/2$ and $Q > 0$ for which

$$\lim_{k\to\infty} k^{\alpha}\left(\frac{2^k\varepsilon_k}{k!}\right) = Q . \tag{4.12}$$

Finally, we include the results of numerical computations of ε_k (cf. (1.5)), using the Remez algorithm. Tabulated below in Table I are $2^k\varepsilon_k/k!$ and $\eta_k^{(k)}$ (cf. (4.1)) for $1 \leqq k \leqq 11$, rounded to six decimal places.

TABLE I

k	$2^k\varepsilon_k/k!$	$\sqrt{k}\,2^k\varepsilon_k/k!$	$k^{1/3}2^k\varepsilon_k/k!$	$\eta_k^{(k)}$	$(2k-\eta_k^{(k)})/k^{1/3}$
1	.556 929	.556 929	.556 929	1.278 466	0.721 534
2	.433 831	.613 530	.546 593	3.009 706	0.785 997
3	.369 345	.639 724	.532 688	4.827 187	0.813 183
4	.327 863	.655 726	.520 450	6.684 493	0.828 717
5	.298 228	.666 859	.509 963	8.565 402	0.838 958
6	.275 655	.675 215	.500 899	10.462 169	0.846 301
7	.257 698	.681 806	.492 959	12.370 440	0.851 865
8	.242 958	.687 190	.485 917	14.287 471	0.856 265
9	.230 569	.691 706	.479 602	16.211 473	0.859 834
10	.219 959	.695 570	.473 886	18.141 145	0.862 804
11	.210 735	.698 930	.468 671	20.075 554	0.865 316

Since $\pi^{-1/2} = 0.564\,190\cdots$, we note that the third column of Table I is in numerical agreement with the first inequality of (4.8), while the fourth column of Table I is in numerical agreement with (4.11). Moreover, since the entries of the third and fourth columns of Table I are respectively strictly increasing and strictly decreasing, it would appear that $1/3 < \alpha < 1/2$ if (4.12) were valid. Also, as a consequence of (4.4), we have

$$2.945\,820\cdots + o(1) > (2k - \eta_k^{(k)})/k^{1/3} \geqq 0\,, \quad \text{as} \quad k \longrightarrow \infty\,, \tag{4.13}$$

which again is in agreement with the last column on Table I.

ACKNOWLEDGMENT. We wish to thank Dr. M. Lachance (University of South Florida) for kindly performing the calculations for Table I.

Added in Proof. Using a different method the authors have recently extended Theorem 2.1 to the case of L^q-norms, $q \geqq 1$, on [0, 1].

References

1. I. Borosh, C. K. Chui, and P. W. Smith, *On approximation of x^N by incomplete polyno- mials*, J. Approximation Theory, **24** (1978), 227-235.

2. D. K. Faddeev and I. S. Sominskii, *Problems in Higher Algebra* (translated by J. L Brenner), W. W. Freeman and Co., San Francisco and London, 1965.

3. M. Lachance, E. B. Saff, and R. S. Varga, *Bounds for incomplete polynomials vanishin at both endpoints of an interval*, Constructive Approaches to Mathematical Models (C. V Coffman and G. J. Fix, eds.), 421–437, Academic Press, Inc., New York, 1979.

4. G. G. Lorentz, *Approximation by incomplete polynomials (problems and resnjts)*, Pad and Rational Approximation: Theory and Applications (E. B. Saff and R. S. Varga, eds. 289-302, Academic Press, Inc., New York, 1977.

5. G. Meinardus, *Approximation of Functions: Theory and Numerical Methods*, Springer Verlag, New York, 1967.

6. E. B. Saff and R. S. Varga, *The sharpness of Lorentz's theorem on incomplete polyno mials*, Trans. Amer. Math. Soc., **249** (1979), 163–186.

7. ———, *On incomplete polynomials*, Numerische Methoden der Approximations theorie, Band 4 (L. Collatz, G. Meinardus, and H. Werner, eds.), pp. 281-298. ISNM vo 42, Birkhäuser Verlag, Basel and Stuttgart, 1978.

8. ———, *Uniform approximation by incomplete polynomials*, Internat. J. Math. an Math. Sci., 1 (1978), 407-420.

9. G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Pub lication Volume XXIII, fourth ed., American Mathematical Society, Providence, Rhod Island, 1975.

Received May 31, 1979. Research by the first author was supported in part by th Air Force Office of Scientific Research, and by a Guggenheim Memorial Foundatio Fellowship. Research by the second author was supported in part by the Air Forc Office of Scientific Research, and by the Department of Energy.

UNIVERSITY OF SOUTH FLORIDA
TAMPA, FL 33620
AND
KENT STATE UNIVESITY
KENT, OH 44242