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On Lacunary Incomplete Polynomials

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1. Introduction

This paper is a continuation of the authors' investigation (cf. [3, 6–10]) of certain classes of polynomials first introduced by Lorentz [4]. These polynomials, called *incomplete polynomials*, have several of their first coefficients equal to zero and are of the following forms:

$$Q(x)=x^s \sum_{i=0}^{k} a_i x^i, \tag{1.1}$$

and (more generally)

$$q(x)=x^s \sum_{i=0}^{k} b_i x^{\mu_i}, \quad 0 \leqq \mu_0 < \mu_1 < \ldots < \mu_k, \quad (\mu_i \text{ integers}), \tag{1.2}$$

where in both forms we assume that $s>0$. For convenience, we call the polynomial in (1.2) a *lacunary incomplete polynomial.* Quite a few recent investigations have been devoted to the study of incomplete polynomials. We refer the reader to a survey article of Lorentz [5] which includes an extensive bibliography.

Our purpose in the present paper is two-fold: (i) to extend several of the known results concerning incomplete polynomials of the form (1.1) to the more general class of lacunary incomplete polynomials (1.2), and (ii) to study the limiting behavior of certain sequences of L_q-extremal incomplete polynomials, where L_q, $1 \leqq q < \infty$, refers to the q-th power integral norm over $[0, 1]$.

To state our contributions more precisely, we first introduce some needed notation. We let π_m denote the class of all polynomials of degree at most m having real coefficients. For each pair (s, k) of nonnegative integers, we denote by $\pi_{s,k}$ the collection of polynomials

$$\pi_{s,k} := \{x^s P(x): P \in \pi_k\}, \tag{1.3}$$

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so that $\pi_{s,k} \subset \pi_{s+k}$. When $s>0$, a polynomial $Q \in \pi_{s,k}$ is called an incomplete polynomial *of type* (s,k).

If I denotes a real (finite or infinite) interval and if h is continuous on I, we set

$$\|h\|_{L_q(I)} := \left(\int_I |h(x)|^q \, dx\right)^{1/q}, \quad 1 \leqq q < \infty, \tag{1.4}$$

$$\|h\|_{L_\infty(I)} := \sup\{|h(x)| : x \in I\}. \tag{1.5}$$

One of the basic properties of incomplete polynomials of the form (1.1) is the following result of the authors [8], which sharpens related work of Lorentz.

Theorem A. *Let s and k be positive integers. If $Q \in \pi_{s,k}, Q \not\equiv 0$, and if ξ is a point in $[0,1]$ such that $|Q(\xi)| = \|Q\|_{L_\infty[0,1]}$, then*

$$\left(\frac{s}{s+k}\right)^2 < \xi. \tag{1.6}$$

We remark that the inequality (1.6) is best possible in a certain limiting sense which is described in detail in [7].

One of our goals in the present paper is to show that the inequality (1.6) holds, more generally, for any lacunary incomplete polynomial of the form (1.2); that is, the conclusion of Theorem A does not depend on the precise degree of the polynomial. Rather, it depends on the order s of the zero at $x=0$ and on the number of nonzero coefficients of the polynomial. To be specific, we shall prove (as a special case of Theorem 2.4 in Sect. 2) the following.

Theorem 1.1. *If $q(x)(\not\equiv 0)$ is a polynomial of the form $q(x) = x^s \left(\sum_{i=0}^{k} b_i x^{\mu_i}\right)$, with $s>0$ and $k>0$, and if $\xi \in [0,1]$ is such that $|q(\xi)| = \|q\|_{L_\infty[0,1]}$, then $[s/(s+k)]^2 < \xi$.*

One of the most elegant results on the limits of sequences of incomplete polynomials of the form (1.1) concerns their "forced convergence to zero." Stated in the version originally proved by Lorentz [4], we have

Theorem B. *Let $\{Q_n(x)\}$ be a sequence of incomplete polynomials of respective types (s_n, k_n), and suppose there exists a number θ, with $0 < \theta \leqq 1$, such that*

$$\theta \leqq s_n/(s_n + k_n), \quad \text{for all } n, \tag{1.7}$$

where $s_n + k_n \to \infty$ as $n \to \infty$. If, moreover, the sequence $\{Q_n(x)\}$ is uniformly bounded on $[0,1]$, then

$$\lim_{n\to\infty} Q_n(x) = 0, \quad \text{for all } 0 \leqq x < \theta^2, \tag{1.8}$$

uniformly on closed subsets of $[0, \theta^2)$.

We remark that the uniform boundedness assumption in Theorem B can be considerably weakened, and furthermore that the convergence to zero in (1.8) holds, in fact, in a region of the complex plane that intersects the nonnegative real axis in the interval $[0, \theta^2)$. These extensions are given in [2] and [7], and are also discussed in [5].

Here we show that Theorem B holds even for *lacunary* incomplete polynomials. As a special case of our Theorem 2.7 in Sect. 2 we shall deduce

Theorem 1.2. *Let $\{q_n(x)\}$ be a sequence of polynomials of the form*

$$q_n(x) = x^{s(n)} \left(\sum_{i=0}^{k(n)} b_i(n)\, x^{\mu_i(n)} \right), \tag{1.9}$$

and suppose there exists a number θ, with $0 < \theta \leqq 1$, such that

$$\theta \leqq s(n)/(s(n)+k(n)), \quad \text{for all } n, \tag{1.10}$$

where $s(n)+k(n) \to \infty$ as $n \to \infty$. If, moreover, the sequence $\{q_n(x)\}$ is uniformly bounded on $[0,1]$, then

$$\lim_{n\to\infty} q_n(x) = 0, \quad \text{for all } 0 \leqq x < \theta^2, \tag{1.11}$$

uniformly on closed subsets of $[0, \theta^2)$.

Much more is actually proved in Sect. 2, where we establish domination theorems for lacunary incomplete polynomials, and also consider weight functions more general than $w(x) = x^s$.

In Sect. 3 we study the following extremal problem with respect to the L_q-norm on $[0,1]$. Let $0 \leqq \mu_0 < \mu_1 < \ldots < \mu_k$ be $k+1$ fixed integers, and for each nonnegative integer n let

$$E_n = E_n(\mu_0, \ldots, \mu_k, q) := \inf\left\{ \left\| x^n \left(x^{\mu_k} - \sum_{j=0}^{k-1} c_j x^{\mu_j} \right) \right\|_{L_q[0,1]} \right\}, \tag{1.12}$$

where the infimum is taken over all $(c_0, c_1, \ldots, c_{k-1}) \in \mathbb{R}^k$, and where $1 \leqq q \leqq \infty$. We find (cf. Theorem 3.1) the precise limiting behavior (as $n \to \infty$) of these errors, E_n, as well as the limiting behavior of the extremal polynomials for the problem (1.12). In particular, we establish that

$$\lim_{n\to\infty} n^{k+1/q} E_n = \frac{\varepsilon_k}{k!} \prod_{j=0}^{k-1} (\mu_k - \mu_j), \tag{1.13}$$

where

$$\varepsilon_k = \varepsilon_k(q) := \inf\{ \| e^{-t}(t^k - h(t)) \|_{L_q[0,+\infty)} : h \in \pi_{k-1} \}. \tag{1.14}$$

The Eq. (1.13) sharpens earlier results of Borosh, Chui, and Smith [1], and extends a previous result of the authors (cf. [10]) for the case $q = \infty$. We remark that when $q = 2$, our results give a generalization of the fact that the classical Laguerre polynomials can be derived as the limit of certain sequences of Jacobi polynomials (cf. [12, p. 103]).

2. Domination Theorems for Incomplete Polynomials

To establish the generalizations stated in Theorems 1.1 and 1.2, we first prove that lacunary incomplete polynomials are dominated by "constrained Chebyshev polynomials" as introduced by the authors in [8]. For further generality, we allow any weight function $w(x)$ on $[0,1]$ which satisfies

$$w\in C[0,1],\ w(0)=0, \quad \text{and} \quad w(x)>0 \quad \text{for } x\in(0,1]. \tag{2.1}$$

In the case of the incomplete polynomials (1.1) or (1.2), this weight function corresponds to x^s.

Throughout this section we shall work exclusively with the L_∞-norm over $[0,1]$, and hence, for brevity, we write

$$\|\cdot\|=\|\cdot\|_{L_\infty[0,1]}.$$

Our first main result is

Theorem 2.1. *Let k be a positive integer, and let*

$$P^*(x)=x^k-\sum_{i=0}^{k-1} c_i^* x^i \tag{2.2}$$

be the unique extremal polynomial for the problem

$$\inf\left\{\left\|w(x)\left(x^k-\sum_{i=0}^{k-1} c_i x^i\right)\right\|:(c_0,c_1,\dots,c_{k-1})\in\mathbb{R}^k\right\}, \tag{2.3}$$

where $w(x)$ satisfies (2.1). Set

$$\xi^*:=\min\{x\in(0,1]:|w(x)P^*(x)|=\|wP^*\|\}. \tag{2.4}$$

Then for any lacunary polynomial of the form $p(x)=\sum_{i=0}^{k} b_i x^{\mu_i}$, with $p(x)$ not identically a constant times $P^*(x)$, there holds

$$|p(x)|<\frac{\|wp\|}{\|wP^*\|}|P^*(x)|, \quad \text{for all } 0<x<\xi^*. \tag{2.5}$$

Furthermore, if $\xi\in(0,1]$ satisfies $|w(\xi)p(\xi)|=\|wp\|$, then

$$\xi^*\leqq\xi. \tag{2.6}$$

The proof requires two lemmas. In the first we compare a lacunary polynomial with the extremal polynomial of the same lacunary form.

Lemma 2.2. *Let $\{\mu_i\}_{i=0}^k$ be $k+1$ integers with $0\leqq\mu_0<\mu_1<\dots<\mu_k$, and let $p^*(x)=x^{\mu_k}-\sum_{i=0}^{k-1} b_i^* x^{\mu_i}$ be the unique extremal polynomial for the problem*

$$\inf\left\{\left\|w(x)\left(x^{\mu_k}-\sum_{i=0}^{k-1} b_i x^{\mu_i}\right)\right\|:(b_0,\dots,b_{k-1})\in\mathbb{R}^k\right\}, \tag{2.7}$$

where $w(x)$ satisfies (2.1). *Set*

$$\xi_0^* := \min\{x \in (0,1] : |w(x)p^*(x)| = \|w p^*\|\}, \tag{2.8}$$

$$\xi_k^* := \max\{x \in (0,1] : |w(x)p^*(x)| = \|w p^*\|\}. \tag{2.9}$$

Then, for any polynomial $p(x)$ of the form $p(x) = \sum_{i=0}^{k} b_i x^{\mu_i}$, with $p(x)$ not identically a constant times $p^*(x)$, there holds

$$|p(x)| < \frac{\|w p\|}{\|w p^*\|} |p^*(x)|, \qquad \text{for all } x \in (0, \xi_0^*) \cup (\xi_k^*, \infty). \tag{2.10}$$

Furthermore, for any $\xi \in (0,1]$ for which $|w(\xi)p(\xi)| = \|w p\|$, there holds

$$\xi_0^* \leqq \xi \leqq \xi_k^*. \tag{2.11}$$

Proof. Because $\operatorname{span}\{w(x)x^{\mu_0}, w(x)x^{\mu_1}, \ldots, w(x)x^{\mu_{k-1}}\}$ is a Haar space on $(0,1]$, there is a unique polynomial solution p^* of (2.7), and we let $0 \leqq x_0 < x_1 < \ldots < x_k \leqq 1$ denote $k+1$ alternation points of wp^* in $[0,1]$. Of course, since $w(0)=0$ by (2.1), then $x_0 > 0$. Moreover we may assume, without loss of generality, that $x_0 = \xi_0^*$ of (2.8). Now, the $k+1$ alternation points $\{x_i\}_{i=0}^k$ imply that $p^*(x)$ has at least k distinct zeros in (x_0, x_k), while from Descartes' Rule of Signs, $p^*(x)$ has at most (and thus precisely) k zeros on $(0, +\infty)$. Moreover, since $p^*(x)$ is monic, then $\operatorname{sgn} p^*(x) = 1$ for all $x \geqq x_k$, which implies that

$$\operatorname{sgn} p^*(x) = (-1)^k, \qquad \text{for all } 0 < x \leqq x_0. \tag{2.12}$$

Next, for any $p(x) = \sum_{i=0}^{k} b_i x^{\mu_i} (\not\equiv 0)$, choose any constant $\gamma > 0$ such that

$$\|\gamma w p\| < \|w p^*\|. \tag{2.13}$$

Because of (2.13) and (2.1), the polynomial $p^*(x) - \gamma p(x)$ necessarily alternates in sign in the $k+1$ points $\{x_i\}_{i=0}^k$, so that $p^*(x) - \gamma p(x)$, by the same reasoning as above, has precisely k zeros $\{\tau_i\}_{i=1}^k$ in $(0, +\infty)$, where

$$x_0 < \tau_1 < \tau_2 < \ldots < \tau_k < x_k. \tag{2.14}$$

Writing

$$p^*(x) - \gamma p(x) = \sum_{i=0}^{k} d_i x^{\mu_i} =: S(x) \prod_{i=1}^{k} (x - \tau_i), \tag{2.15}$$

then S is a polynomial (of degree at most $\mu_k - k$) which is nonzero for all $x > 0$. We claim, in fact, that

$$\operatorname{sgn} S(x) = +1, \qquad \text{for all } x > 0. \tag{2.16}$$

To see this, we multiply (2.15) by $(-1)^k w(x)$ and evaluate at $x = x_0$ to obtain

$$\begin{aligned}(-1)^k \gamma w(x_0) p(x_0) &= (-1)^k w(x_0) p^*(x_0) \\ &\quad + (-1)^{k+1} w(x_0) S(x_0) \prod_{i=1}^{k} (x_0 - \tau_i).\end{aligned} \tag{2.17}$$

From (2.12), we have $(-1)^k w(x_0)\,p^*(x_0)=\|w\,p^*\|$, and hence, by our choice of γ in (2.13), the last term in (2.17) must be negative. But $w(x_0)>0$ and $x_0<\tau_i$ for all i (cf. (2.14)), so that necessarily $S(x_0)>0$. This proves (2.16).

Now from (2.12), (2.15), and (2.16), we have

$$(-1)^k\,\gamma\,p(x)<(-1)^k\,p^*(x)=|p^*(x)|, \quad \text{for all } 0<x\leqq x_0.$$

Since we can replace $p(x)$ by $-p(x)$ and not change (2.13), then

$$(-1)^{k+1}\,\gamma\,p(x)<|p^*(x)|, \quad \text{for all } 0<x\leqq x_0;$$

whence

$$|\gamma\,p(x)|<|p^*(x)|, \quad \text{for all } 0<x\leqq x_0. \tag{2.18}$$

On letting γ increase in (2.13) to $\tilde{\gamma}$, where

$$\tilde{\gamma}:=\|w\,p^*\|/\|w\,p\|, \tag{2.19}$$

it follows from (2.18) and continuity that

$$|p(x)|\leqq|p^*(x)|/\tilde{\gamma}, \quad \text{for all } 0\leqq x\leqq x_0=\xi_0^*. \tag{2.20}$$

The same reasoning shows that (2.20) is also valid for $x\geqq x_k$ and hence for $x\geqq\xi_k^*$ (cf. (2.9)). Consequently,

$$|p(x)|\leqq|p^*(x)|/\tilde{\gamma}, \quad \text{for all } x\in[0,\xi_0^*]\cup[\xi_k^*,\infty). \tag{2.21}$$

Now, suppose that equality holds in (2.21) for some $\hat{x}\in(0,\xi_0^*)\cup(\xi_k^*,\infty)$. Without loss of generality, we may assume that

$$p^*(\hat{x})-\tilde{\gamma}\,p(\hat{x})=0. \tag{2.22}$$

From the preceding argument, we know that for each γ with $0<\gamma<\tilde{\gamma}$, the polynomial $p^*(x)-\gamma\,p(x)$ does not vanish for any $x\in(0,\xi_0^*]\cup[\xi_k^*,\infty)$. Hence, again letting γ increase to $\tilde{\gamma}$, either $p^*(x)-\tilde{\gamma}\,p(x)$ is never zero on $(0,\xi_0^*)\cup(\xi_k^*,\infty)$ or it is identically zero. The assumption (2.22) therefore implies that $p^*(x)-\tilde{\gamma}\,p(x)\equiv 0$, which contradicts the hypothesis of Lemma 2.2. Thus, strict inequality holds in (2.21) for all $x\in(0,\xi_0^*)\cup(\xi_k^*,\infty)$, which establishes (2.10).

Finally, if $\xi\in(0,1]$ is such that $|w(\xi)\,p(\xi)|=\|w\,p\|$, a short calculation with (2.10) directly shows that ξ cannot lie in $(0,\xi_0^*)$ or in $(\xi_k^*,1]$ if $\xi_k^*<1$; whence $\xi_0^*\leqq\xi\leqq\xi_k^*$, which gives (2.11) of Lemma 2.2. □

Our next lemma will enable us to prove Theorem 2.1 by means of an inductive argument.

Lemma 2.3. *Let* $\{\mu_i(1)\}_{i=0}^k$ *and* $\{\mu_i(2)\}_{i=0}^k$ *be two sets of* $k+1$ *integers, where* $0\leqq\mu_0(j)<\mu_1(j)<\ldots<\mu_k(j)$ *for* $j=1,2$. *Suppose further that* $\mu_k(1)>\mu_k(2)$ *and that* $\{\mu_i(1)\}_{i=0}^k\cup\{\mu_i(2)\}_{i=0}^k$ *has at most* $k+2$ *distinct elements. For* $j=1,2$, *let*

$$p_j^*(x)=x^{\mu_k(j)}-\sum_{i=0}^{k-1}b_i^*(j)\,x^{\mu_i(j)} \tag{2.23}$$

be the unique extremal polynomial for the problem of (2.7), *where* $\mu_i = \mu_i(j)$, *and where* $w(x)$ *satisfies* (2.1). *Set*

$$\xi_0^*(j) := \min\{x \in (0,1] : |w(x)\, p_j^*(x)| = \|w\, p_j^*\|\}, \qquad j = 1, 2. \tag{2.24}$$

Then,

$$|p_1^*(x)| < \frac{\|w\, p_1^*\|}{\|w\, p_2^*\|} |p_2^*(x)|, \qquad \textit{for all } 0 < x < \xi_0^*(2). \tag{2.25}$$

Moreover,

$$\xi_0^*(2) \leqq \xi_0^*(1). \tag{2.26}$$

Proof. For $j = 1, 2$ we let

$$0 < \xi_0^*(j) = x_0^{(j)} < x_1^{(j)} < \ldots < x_k^{(j)} \leqq 1$$

be $k+1$ alternation points of $w\, p_j^*$ in $(0,1]$, and we note that p_j^* has precisely k zeros $\{y_i^{(j)}\}_{i=1}^k$ in $(0, +\infty)$, where $x_0^{(j)} < y_1^{(j)} < \ldots < y_k^{(j)} < x_k^{(j)}$. Since each p_j^* is monic, we therefore have

$$\operatorname{sgn} p_j^*(x) = \begin{cases} +1, & \text{for } x \in [x_k^{(j)}, +\infty), \\ (-1)^k, & \text{for } x \in (0, x_0^{(j)}]. \end{cases} \tag{2.27}$$

Now choose any $\gamma > 0$ for which

$$\|\gamma\, w\, p_1^*\| < \|w\, p_2^*\|. \tag{2.28}$$

As in the proof of Lemma 2.2, we deduce from (2.28) that $p_2^*(x) - \gamma\, p_1^*(x)$ has at least k distinct zeros $\{\tau_i\}_{i=1}^k$ with $x_0^{(2)} < \tau_i < x_k^{(2)}$ for all i. Also, since $\mu_k(1) > \mu_k(2)$ and since $\gamma > 0$, then $p_2^*(x) - \gamma\, p_1^*(x) \to -\infty$ as $x \to \infty$. On the other hand, since $w(x_k^{(2)})\, p_2^*(x_k^{(2)}) = \|w\, p_2^*\|$ from (2.27), inequality (2.28) implies that

$$w(x_k^{(2)})\, p_2^*(x_k^{(2)}) - \gamma\, w(x_k^{(2)})\, p_1^*(x_k^{(2)}) \geqq \|w\, p_2^*\| - \|\gamma\, w\, p_1^*\| > 0.$$

Consequently, there is a point $\tau_{k+1} > x_k^{(2)}$ for which

$$p_2^*(\tau_{k+1}) - \gamma\, p_1^*(\tau_{k+1}) = 0.$$

Thus $p_2^*(x) - \gamma\, p_1^*(x)$ has at least $k+1$ distinct zeros in $x > x_0^{(2)}$. But, by hypothesis, this polynomial has at most $k+2$ nonzero coefficients, and so Descartes' Rule of Signs implies that $p_2^*(x) - \gamma\, p_1^*(x)$ has precisely $k+1$ zeros on $(0, +\infty)$, namely at the points $\{\tau_i\}_{i=1}^{k+1}$ with $\tau_i > x_0^{(2)}$ for all i.

Hence we can write

$$p_2^*(x) - \gamma\, p_1^*(x) = T(x) \prod_{i=1}^{k+1} (x - \tau_i), \tag{2.29}$$

where $T(x)$ is a polynomial which is nonzero for all $x > 0$. Clearly, since $p_2^*(x) - \gamma\, p_1^*(x) \to -\infty$ as $x \to \infty$, we must have $\operatorname{sgn} T(x) = -1$ for all $x > 0$. Consequently,

$$(-1)^k \gamma\, p_1^*(x) < (-1)^k p_2^*(x), \qquad \text{for all } 0 < x < \tau_1,$$

and so from the sign properties (2.27) and the fact that $\tau_1 > x_0^{(2)}$, we have

$$\gamma\,|p_1^*(x)| < |p_2^*(x)|,$$

for all

$$0 < x \leqq \min(x_0^{(1)}, x_0^{(2)}) = \min(\xi_0^*(1), \xi_0^*(2)).$$

On letting γ increase to its supremum value, viz. $\hat{\gamma} := \|w\,p_2^*\|/\|w\,p_1^*\|$, in (2.28), the last inequality becomes

$$|p_1^*(x)| \leqq \frac{\|w\,p_1^*\|}{\|w\,p_2^*\|}\,|p_2^*(x)|, \qquad \text{for all } 0 \leqq x \leqq \min(\xi_0^*(1), \xi_0^*(2)). \tag{2.30}$$

Now, if we assume that $\xi_0^*(1) < \xi_0^*(2)$, then multiplying (2.30) by $w(x)$ and evaluating at $x = \xi_0^*(1)$ gives

$$\|w\,p_1^*\| \leqq \frac{\|w\,p_1^*\|}{\|w\,p_2^*\|}\,|w(\xi_0^*(1))\,p_2^*(\xi_0^*(1))| < \|w\,p_1^*\|,$$

the last inequality coming from the definition of $\xi_0^*(2)$. As the above is absurd, then $\xi_0^*(2) \leqq \xi_0^*(1)$, which is the desired inequality of (2.26).

Finally, we observe that (2.30) is valid for all $0 \leqq x \leqq \xi_0^*(2)$ and, by reasoning as in the proof of Lemma 2.2, strict inequality must hold in (2.30) for all $x \in (0, \xi_0^*(2))$. □

We can now give the

Proof of Theorem 2.1. First note that if $\mu_i = i$ for all $i = 0, 1, \ldots, k$, then (2.5) and (2.6) follow directly from (2.10) and (2.11) of Lemma 2.2. Hence we assume that $\{\mu_i\}_{i=0}^k \neq \{i\}_{i=0}^k$. For convenience, we set $\mu_i^{(1)} := \mu_i$, $i = 0, 1, \ldots, k$, and we recursively define the sets of integers $\{\mu_i^{(r)}\}_{i=0}^k$, $r = 2, 3, \ldots$, as follows.

Set

$$J_r := \{i : 0 \leqq i \leqq k-1 \text{ and } \mu_{i+1}^{(r)} - \mu_i^{(r)} \geqq 2\}, \qquad r = 1, 2, \ldots. \tag{2.31}$$

If $J_r \neq \varnothing$, set

$$\lambda(r) := \max\{i : i \in J_r\} \tag{2.32}$$

and define

$$\mu_i^{(r+1)} := \begin{cases} \mu_i^{(r)}, & \text{for } i = 0, 1, \ldots, \lambda(r), \\ \mu_i^{(r)} - 1, & \text{for } i = \lambda(r)+1, \ldots, k. \end{cases} \tag{2.33}$$

If $J_r = \varnothing$ and $\mu_0^{(r)} > 0$, set $\mu_i^{(r+1)} := \mu_i^{(r)} - 1$, $i = 0, 1, \ldots, k$. It is easily seen that if $J_r \neq \varnothing$ or if $\mu_0^{(r)} > 0$, then $0 \leqq \mu_0^{(r+1)} < \mu_1^{(r+1)} < \ldots < \mu_k^{(r+1)}$, and moreover the two sets $\{\mu_i^{(r)}\}_{i=0}^k$, $\{\mu_i^{(r+1)}\}_{i=0}^k$ satisfy the hypotheses of Lemma 2.3. Consequently, if $p_r^*(x)$ and $p_{r+1}^*(x)$ are the extremal polynomials (cf. (2.7)) respectively associated with $\{\mu_i^{(r)}\}_{i=0}^k$ and $\{\mu_i^{(r+1)}\}_{i=0}^k$, and if $\xi_0^*(r)$ and $\xi_0^*(r+1)$ are the corresponding extreme points as defined in (2.24), then it follows from Lemma 2.3 that

$$\xi_0^*(r+1) \leqq \xi_0^*(r), \tag{2.34}$$

and also that

$$|p_r^*(x)| < \frac{\|w p_r^*\|}{\|w p_{r+1}^*\|} |p_{r+1}^*(x)|, \qquad \text{for all } 0<x<\xi_0^*(r+1). \tag{2.35}$$

The essential point of the above construction is that if $J_r \neq \varnothing$, then the length of the last gap in $\{\mu_i^{(r)}\}_{i=0}^k$, namely $\mu_{\lambda(r)+1}^{(r)} - \mu_{\lambda(r)}^{(r)}$, is reduced by unity in passing to $\{\mu_i^{(r+1)}\}_{i=0}^k$. Obviously, by continuing this procedure, we will arrive, in say M steps, at the point where $J_M = \varnothing$ and $\mu_0^{(M)} = 0$, that is $\{\mu_i^{(M)}\}_{i=0}^k = \{i\}_{i=0}^k$. Thus, in the notation of Theorem 2.1, $p_M^*(x) = P^*(x)$ and $\xi_0^*(M) = \xi^*$, and, moreover, the inequalities (2.34) and (2.35) imply that

$$\xi^* \leqq \xi_0^*(1), \tag{2.36}$$

and that

$$|p_1^*(x)| < \frac{\|w p_1^*\|}{\|w P^*\|} |P^*(x)|, \qquad \text{for all } 0<x<\xi^*. \tag{2.37}$$

Finally, if $p(x)$ is any lacunary polynomial of the form $p(x) = \sum_{i=0}^{k} b_i x^{\mu_i} \not\equiv 0$, then it follows from Lemma 2.2 that

$$|p(x)| \leqq \frac{\|w p\|}{\|w p_1^*\|} |p_1^*(x)|, \qquad \text{for all } 0<x<\xi_0^*(1).$$

Combining the above inequality with (2.37) gives the desired result (2.5) of Theorem 2.1 since, from (2.36), we have $\xi^* \leqq \xi_0^*(1)$. Similarly, for any $\xi \in (0,1]$ for which $|w(\xi)p(\xi)| = \|wp\|$, we have $\xi_0^*(1) \leqq \xi$ from (2.11) of Lemma 2.2, which combined with (2.36), gives the desired result (2.6). □

We remark that if the weight function $w(x)$ is differentiable at $x = \xi^*$, then it is easy to show that *strict* inequality must hold in (2.6) of Theorem 2.1.

We now give an interesting consequence of Theorem 2.1. For this purpose, we use the (nonnormalized) *constrained Chebyshev polynomials* $Q_{s,k}(x)$ studied by the authors in [8]. Namely, for each pair of positive integers (s,k), $Q_{s,k}(x)$ is the unique incomplete polynomial of the form

$$Q_{s,k}(x) = x^s \left(x^k - \sum_{i=0}^{k-1} a_i^* x^i \right) \tag{2.38}$$

such that

$$\|Q_{s,k}\| = \inf \left\{ \left\| x^s \left(x^k - \sum_{i=0}^{k-1} a_i x^i \right) \right\| : (a_0, a_1, \ldots, a_{k-1}) \in \mathbb{R}^k \right\}. \tag{2.39}$$

Further, following the notation of [8], we let $\xi_0^{(s,k)}$ denote the least alternation point on $(0,1]$ for $Q_{s,k}(x)$. Then from Theorem A of the introduction (cf. [8, Prop. 7]), we know that

$$\left(\frac{s}{s+k} \right)^2 < \xi_0^{(s,k)}. \tag{2.40}$$

Applying Theorem 2.1 we can now easily deduce

Theorem 2.4. *Suppose that* $w(x)=g(x)x^s$, *where* $s>0$ *is an integer, and where* $g(x)$ *is continuous on* $[0,1]$, *and is positive and nondecreasing on* $(0,1]$. *Let* $\{\mu_i\}_{i=0}^{k}$ *be any set of* $k+1$ *integers with* $0\leqq\mu_0<\mu_1<\ldots<\mu_k$, *and let* $p(x)$ *be any polynomial of the form* $p(x)=\sum_{i=0}^{k} b_i x^{\mu_i}$, *with* $x^s p(x)$ *not identically a constant times* $Q_{s,k}(x)$. *Then*

$$|w(x)p(x)|<\frac{\|w\,p\|}{\|Q_{s,k}\|}|Q_{s,k}(x)|, \qquad \text{for all } 0<x<\xi_0^{(s,k)}. \tag{2.41}$$

Furthermore, if ξ *is any point in* $(0,1]$ *such that* $|w(\xi)p(\xi)|=\|w\,p\|$, *then*

$$\left(\frac{s}{s+k}\right)^2<\xi_0^{(s,k)}\leqq\xi. \tag{2.42}$$

Proof. Let $\sigma:=\min\{x\in(0,1]:|x^s p(x)|=\|x^s p(x)\|\}$. Then from (2.5) and (2.6) of Theorem 2.1 (with weight function x^s) we know that

$$|x^s p(x)|<\frac{\|x^s p(x)\|}{\|Q_{s,k}\|}|Q_{s,k}(x)|, \qquad \text{for all } 0<x<\xi_0^{(s,k)}, \tag{2.43}$$

and that $\xi_0^{(s,k)}\leqq\sigma$. Hence, on multiplying (2.43) by $g(x)$ and using the hypothesis that $g(x)$ is positive and nondecreasing on $(0,1]$, we have

$$|w(x)p(x)|<g(\sigma)\frac{\|x^s p(x)\|}{\|Q_{s,k}\|}|Q_{s,k}(x)|, \qquad \text{for all } 0<x<\xi_0^{(s,k)}. \tag{2.44}$$

But

$$\|w\,p\|=\|g(x)x^s p(x)\|\geqq g(\sigma)|\sigma^s p(\sigma)|=g(\sigma)\|x^s p(x)\|,$$

and so (2.41) follows from (2.44). The inequality (2.42) is an immediate consequence of the inequalities (2.40) and (2.44). □

We remark that on taking $w(x)=x^s$ in Theorem 2.4 we deduce the result of Theorem 1.1 announced in the introduction. As other immediate consequences of Theorem 2.4 we have

Corollary 2.5. *For any pair* (r,m) *of positive integers, there holds*

$$\xi_0^{(r,m)}<\xi_0^{(r+1,m)}; \qquad \xi_0^{(r,m+1)}<\xi_0^{(r,m)}, \tag{2.45}$$

where $\xi_0^{(s,k)}$ *is the least alternation point for* $Q_{s,k}(x)$ *in* $(0,1]$.

Corollary 2.6. *Let* $w(x)$ *be as in Theorem 2.4, and let* $p^*(x)=x^{\mu_k}-\sum_{i=0}^{k-1} b_i^* x^{\mu_i}$ *be the extremal polynomial for the problem of* (2.7). *Then all the positive real zeros of* $p^*(x)$ *are in the interval* $[s/(s+k)]^2<x<1$.

To state the final result of this section we shall define a function G which majorizes the constrained Chebyshev polynomials $Q_{s,k}$ of (2.38). For each fixed

θ, with $0<\theta<1$, let

$$z=\psi(w)=\psi(w;\theta):=\frac{1+\theta^2}{2}+\frac{1-\theta^2}{2}\left\{\frac{w+w^{-1}}{2}\right\}$$

map the exterior of the circle $|w|=1$ in the w-plane onto the exterior of the interval $[\theta^2,1]$ in the z-plane, and let $w=\phi(z)$ denote the inverse of ψ. Then we set

$$G(z;\theta):=|\phi(z)|\left|\frac{(1-\theta)\,\phi(z)+(1+\theta)}{(1+\theta)\,\phi(z)+(1-\theta)}\right|^{\theta}, \quad \text{for all } z\in\mathbb{C}\setminus[\theta^2,1], \tag{2.46}$$

and observe that $G(0;\theta)=0$, and that $0\leqq G(x;\theta)<1$ for all $0\leqq x<\theta^2$. It was proved by Kemperman and Lorentz [2] and by Saff and Varga [7] that, for each pair of positive integers (s,k), we have

$$|Q_{s,k}(z)|\leqq\|Q_{s,k}\|\left[G\left(z;\frac{s}{s+k}\right)\right]^{s+k}, \quad \text{for all } z\in\mathbb{C}\setminus\left[\left(\frac{s}{s+k}\right)^2,1\right]. \tag{2.47}$$

Using this fact, we prove

Theorem 2.7. *Let $\{q_n(x)\}$ be a sequence of polynomials of the form*

$$q_n(x)=x^{s(n)}\left(\sum_{i=0}^{k(n)} b_i(n)\,x^{\mu_i(n)}\right),$$

where $s(n)>0$, $k(n)>0$ for all n, and where $s(n)+k(n)\to\infty$ as $n\to\infty$. Suppose that there exists a constant θ, with $0<\theta<1$, such that

$$\theta\leqq\frac{s(n)}{s(n)+k(n)}, \quad \text{for all } n=1,2,\ldots. \tag{2.48}$$

If

$$\limsup_{n\to\infty}\|q_n\|^{\frac{1}{s(n)+k(n)}}\leqq 1, \tag{2.49}$$

then

$$\lim_{n\to\infty} q_n(x)=0, \quad \text{for all } 0\leqq x<\theta^2,$$

uniformly on closed subsets of $[0,\theta^2)$. Moreover, this convergence is geometric, in the sense that for any closed set $K\subset[0,\theta^2)$ there holds

$$\limsup_{n\to\infty}[\max|q_n(x)|;x\in K]^{\frac{1}{s(n)+k(n)}}\leqq\max\{G(x;\theta):x\in K\}<1. \tag{2.50}$$

Proof. From (2.41) of Theorem 2.4 (with $w(x)=x^{s(n)}$) we have that

$$|q_n(x)|\leqq\frac{\|q_n\|}{\|Q_{s(n),k(n)}\|}|Q_{s(n),k(n)}(x)|, \quad \text{for all } 0\leqq x\leqq\xi_0^{(s(n),k(n))}.$$

Using this together with (2.42) and (2.47) gives

$$|q_n(x)| \leqq \|q_n\| \left[G\left(x; \frac{s(n)}{s(n)+k(n)}\right)\right]^{s(n)+k(n)},$$

$$\text{for all } 0 \leqq x \leqq \left(\frac{s(n)}{s(n)+k(n)}\right)^2.$$

and so, from (2.48) and the monotonic properties of G (cf. [7, Lemma 4.3]), we have

$$|q_n(x)| \leqq \|q_n\| \, [G(x;\theta)]^{s(n)+k(n)}, \quad \text{for all } 0 \leqq x \leqq \theta^2. \tag{2.51}$$

Hence, on taking the $(s(n)+k(n))$th root of (2.51) and using the hypothesis (2.49), the desired result of (2.50) follows. □

3. An L_q-Extremal Problem for Incomplete Polynomials

In this section, we investigate the problem of best approximation to x^N in the L_q-norm on $[0,1]$ by lacunary incomplete polynomials. Our primary result is

Theorem 3.1. *Let the* $k+1$ *integers* $\mu_0, \mu_1, \ldots, \mu_k$ *be fixed, with* $0 \leqq \mu_0 < \mu_1 < \ldots < \mu_k$. *For each nonnegative integer* n, *set*

$$E_n = E_n(\mu_0, \ldots, \mu_k, q) := \inf\left\{\left\| x^n \left(x^{\mu_k} - \sum_{j=0}^{k-1} c_j x^{\mu_j}\right)\right\|_{L_q[0,1]}\right\}, \tag{3.1}$$

where the infimum is taken over all $(c_0, c_1, \ldots, c_{k-1}) \in \mathbb{R}^k$, *and where* $1 \leqq q \leqq \infty$. *Then*

$$\lim_{n\to\infty} n^{k+1/q} E_n = \frac{\varepsilon_k}{k!} \prod_{j=0}^{k-1} (\mu_k - \mu_j), \tag{3.2}$$

where

$$\varepsilon_k = \varepsilon_k(q) := \inf\{\|e^{-t}(t^k - h(t))\|_{L_q[0,+\infty)} : h \in \pi_{k-1}\}. \tag{3.3}$$

Furthermore, if $\hat{p}_n(x) = \hat{p}_n(x; \mu_0, \ldots, \mu_k, q)$ *is the unique polynomial of the form* $\hat{p}_n(x) = x^{\mu_k} - \sum_{j=0}^{k-1} \hat{c}_j(n) x^{\mu_j}$ *such that*

$$\|x^n \hat{p}_n(x)\|_{L_q[0,1]} = E_n, \quad \text{for all } n \geqq 0, \tag{3.4}$$

and if $\tilde{p}(t)$ *is the unique monic polynomial of degree* k *such that*

$$\|e^{-t} \tilde{p}(t)\|_{L_q[0,+\infty)} = \varepsilon_k, \tag{3.5}$$

then

$$\lim_{n\to\infty} n^k \hat{p}_n\left(1 - \frac{t}{n}\right) = \tilde{p}(t) \frac{(-1)^k}{k!} \prod_{j=0}^{k-1} (\mu_k - \mu_j), \quad \text{for all } t \in \mathbb{R}, \tag{3.6}$$

uniformly on each compact set in $\mathbb{R}$.

Theorem 3.1 considerably sharpens an earlier result due to Borosh, Chui, and Smith [1, Theorem 3]. They proved that there exist positive constants $\sigma_i = \sigma_i(\mu_0, \ldots, \mu_k, q)$, $i=1,2$, such that

$$\sigma_1 \leqq n^{k+1/q} E_n \leqq \sigma_2, \qquad \text{for all } n \geqq 0,$$

and that the coefficients of the extremal polynomials $\hat{p}_n(x)$ are bounded as a function of n. The case $q=\infty$ of Theorem 3.1 was proved by the authors in [10], but the method used there does not immediately extend to the case of finite q. What is crucial in our proof is an inequality for Descartes systems due to Smith [11] (see also [5]).

It is convenient to first state the following simple lemma which, while generalizing results of [10, Lemma 3 and Eq. (2.16)], has the same proof as given there. In this statement, $C(\mu_j, i)$ denotes the binomial coefficient $\binom{\mu_j}{i}$, with the usual convention that $C(\mu_j, i)=0$ if $\mu_j<i$. Further, we set

$$\gamma := \frac{1}{k!} \prod_{j=0}^{k-1} (\mu_k - \mu_j). \tag{3.7}$$

Lemma 3.2. *Let $\{p_n(x)\}$ be any sequence of monic polynomials of the form $p_n(x) = x^{\mu_k} - \sum_{j=0}^{k-1} c_j(n) x^{\mu_j}$, where each p_n has k distinct zeros on $(0, +\infty)$, viz., $0<y_1(n)<y_2(n)<\ldots<y_k(n)$, and where $\lim_{n\to\infty} y_j(n)=1$ for each $j=1,2,\ldots,k$. Writing*

$$p_n(x) = V_n(x) \prod_{j=1}^{k} (x-y_j(n)), \tag{3.8}$$

where $V_n(x)$ is monic of degree $\mu_k - k$ for all n, then

$$\lim_{n\to\infty} V_n\left(1-\frac{t}{n}\right) = \gamma, \qquad \text{(cf. (3.7))} \tag{3.9}$$

uniformly on every compact set in t of $\mathbb{R}$. Furthermore,

$$\lim_{n\to\infty} V_n(x) = \frac{1}{(x-1)^k D} \begin{vmatrix} C(\mu_0,0) & C(\mu_1,0) & \cdots & C(\mu_k,0) \\ \vdots & \vdots & & \vdots \\ C(\mu_0,k-1) & C(\mu_1,k-1) & \cdots & C(\mu_k,k-1) \\ x^{\mu_0} & x^{\mu_1} & \cdots & x^{\mu_k} \end{vmatrix}, \tag{3.10}$$

uniformly on each compact set in x of $\mathbb{R}$, where

$$D := \det[d_{i,j}], \quad d_{i,j} := C(\mu_{j-1}, i-1), \quad i,j=1,2,\ldots,k.$$

We remark that the right side of (3.10) is also a monic polynomial of degree $\mu_k - k$.

We now give the

Proof of Theorem 3.1. As stated earlier, we may assume that $1 \leqq q < \infty$. It is known (cf. [13, p. 64]) that the extremal polynomials $\hat{p}_n(x)$ of (3.4), and $\tilde{p}(t)$ of (3.5) exist and are unique.

Now (3.4) implies that

$$E_n^q=\int_0^1 x^{nq}\,|\hat{p}_n(x)|^q\,dx, \qquad \text{for all } n\geqq 0.$$

On making the change of variables $x=1-t/n$, $t\in[0,n]$, the above equation becomes

$$n^{kq+1}\,E_n^q=\int_0^n \left(1-\frac{t}{n}\right)^{nq}\left|n^k\,\hat{p}_n\left(1-\frac{t}{n}\right)\right|^q dt, \tag{3.11}$$
$$\text{for all } n\geqq 1.$$

Next, we observe that $\hat{p}_n(x)$ has, by Descartes' Rule of Signs, at most k zeros on $(0,+\infty)$, counting multiplicities, while on the other hand, from the orthogonal-like characterization of best weighted (the weight function being x^n) $L_q[0,1]$-approximation to x^{μ_k} (cf. [13, p. 64]), it follows that $\hat{p}_n(x)$ has at least (and thus precisely) k simple zeros in (0,1). Denoting these zeros of $\hat{p}_n(x)$ by

$$0<\hat{x}_1(n)<\hat{x}_2(n)<\ldots<\hat{x}_k(n)<1,$$

we can write

$$\hat{p}_n(x)=S_n(x)\prod_{j=1}^{k}(x-\hat{x}_j(n)), \tag{3.12}$$

where $S_n(x)$ is a monic polynomial of degree μ_k-k which has no zeros on $(0,+\infty)$.

Next, for any polynomial $T(x)$ of the form

$$T(x)=x^{\mu_k}-\sum_{j=0}^{k-1}c_j\,x^{\mu_j}, \tag{3.13}$$

it follows from the definition of E_n in (3.1) that (cf. (3.11))

$$n^{kq+1}\,E_n^q\leqq\int_0^n\left(1-\frac{t}{n}\right)^{nq}\left|n^k\,T\left(1-\frac{t}{n}\right)\right|^q dt.$$

Since $(1-t/n)^{nq}\leqq e^{-tq}$ for all $t\in[0,n]$, we therefore have

$$n^{kq+1}\,E_n^q\leqq\int_0^n e^{-tq}\left|n^k\,T\left(1-\frac{t}{n}\right)\right|^q dt. \tag{3.14}$$

Our immediate goal is to construct specific polynomials of the form (3.13) whose zeros are related to those of $\tilde{p}(t)$. As before, the characterization of best weighted (the weight function now being e^{-t}) $L_q[0,+\infty)$-approximation to t^k by elements in π_{k-1}, gives us that the polynomial $\tilde{p}(t)$ of (3.5) has precisely k simple zeros in $(0,+\infty)$. Calling these zeros (which are independent of n) $0<\tilde{t}_k<\ldots<\tilde{t}_1$, then we can express $\tilde{p}(t)$ as

$$\tilde{p}(t)=\prod_{i=1}^{k}(t-\tilde{t}_i). \tag{3.15}$$

Setting

$$x_i(n):=1-\frac{\tilde{t}_i}{n}, \qquad i=1,2,\ldots,k, \qquad \text{for all } n\geqq 1, \tag{3.16}$$

then all the $x_i(n)$'s are positive for every $n>\tilde{t}_1$. Since the functions $\{x^{\mu_j}\}_{j=0}^{k}$ form a Haar system on $(0,+\infty)$, there exists, for each $n>\tilde{t}_1$, a unique monic polynomial $T_n(x)$ of the form (3.13) which vanishes in each of the points $x_i(n)$, $1\leqq i\leqq k$. On writing

$$T_n(x)=U_n(x)\prod_{i=1}^{k}(x-x_i(n)), \tag{3.17}$$

where $U_n(x)$ is monic of degree μ_k-k, and on substituting $T_n(x)$ in (3.14), we deduce from (3.15) and (3.16) that

$$n^{kq+1}E_n^q\leqq\int_0^n e^{-tq}\left|U_n\left(1-\frac{t}{n}\right)\right|^q|\tilde{p}(t)|^q\,dt, \qquad \text{for all } n>\tilde{t}_1. \tag{3.18}$$

Observe that, since the numbers $\tilde{t}_i$ of (3.15) are independent of n, then the $x_i(n)$ of (3.16) tend to unity as $n\to\infty$, for each $i=1,2,\ldots,k$. Consequently, Lemma 3.2 may be applied to the particular sequence $T_n(x)$ of (3.17). On choosing $[0,1]$ to be the compact set in x in (3.10), and on applying the Lebesgue Dominated Convergence Theorem to the integral of (3.18), it follows from (3.9) that

$$\overline{\lim_{n\to\infty}}\, n^{kq+1}E_n^q\leqq\gamma^q\int_0^\infty e^{-tq}|\tilde{p}(t)|^q\,dt=\gamma^q\varepsilon_k^q, \tag{3.19}$$

the last equality following from (3.5).

Returning to the representation for $\hat{p}_n(x)$ in (3.12), the numbers $\hat{x}_j(n)$ then generate the numbers $t_j(n)$, where

$$t_j(n):=n(1-\hat{x}_j(n)), \qquad \text{for all } 1\leqq j\leqq k, \qquad \text{for all } n\geqq 1. \tag{3.20}$$

We claim that there exists a positive constant M such that

$$|t_j(n)|\leqq M, \qquad \text{for all } 1\leqq j\leqq k, \qquad \text{for all } n\geqq 1. \tag{3.21}$$

To see this, we have from (3.11), (3.12), and (3.20) that

$$n^{kq+1}E_n^q=\int_0^n\left(1-\frac{t}{n}\right)^{nq}\left|S_n\left(1-\frac{t}{n}\right)\right|^q\prod_{j=1}^{k}|t-t_j(n)|^q\,dt, \tag{3.22}$$
$$\text{for all } n\geqq 1.$$

Furthermore, from (3.19), there is an $n_0>1$ such that $n^{kq+1}E_n^q\leqq\gamma^q\varepsilon_k^q+1$ for all $n\geqq n_0$; whence

$$\int_0^n\left(1-\frac{t}{n}\right)^{nq}\left|S_n\left(1-\frac{t}{n}\right)\right|^q\prod_{j=1}^{k}|t-t_j(n)|^q\,dt$$
$$\leqq\gamma^q\varepsilon_k^q+1=:A, \qquad \text{for all } n\geqq n_0.$$

On reducing the interval of integration to $0 \leqq t \leqq 1$, and on bounding below the first term of the integrand, this implies that

$$\left(1-\frac{1}{n}\right)^{nq} \int_0^1 \left|S_n\left(1-\frac{t}{n}\right)\right|^q \prod_{j=1}^{k} |t-t_j(n)|^q \, dt \leqq A.$$

Since $\left(1-\frac{1}{n}\right)^{nq} \to e^{-q}$ as $n \to \infty$, it then follows that there is a constant A' such that

$$\int_0^1 \left|S_n\left(1-\frac{t}{n}\right)\right|^q \prod_{j=1}^{k} |t-t_j(n)|^q \, dt \leqq A', \qquad \text{for all } n \geqq 1. \tag{3.23}$$

Next, we wish to compare the polynomial $\hat{p}_n(x) = S_n(x) \prod_{j=1}^{k} (x - \hat{x}_j(n))$ with the specific monic polynomial

$$H_n(x) := x^{\mu_k - k} \prod_{j=1}^{k} (x - \hat{x}_j(n)) = \sum_{i=0}^{k} b_i(n)\, x^{\mu_k - k + i}. \tag{3.24}$$

Notice that the exponents $\lambda_i := \mu_k - k + i$, $i = 0, 1, \ldots, k$, of the terms in $H_n(x)$ satisfy $\lambda_i \geqq \mu_i$, for all $0 \leqq i \leqq k$, where the μ_i are the corresponding exponents in $\hat{p}_n(x)$. Furthermore, both $\hat{p}_n(x)$ and $H_n(x)$ are monic of degree μ_k, and both vanish in the k points $0 < \hat{x}_1(n) < \ldots < \hat{x}_k(n) < 1$. Hence, as the functions $\{x^j\}_{j=0}^{\mu_k}$ form a Descartes system on $(0, +\infty)$, Theorem 1 of Smith [11] implies that

$$|H_n(x)| \leqq |\hat{p}_n(x)|, \qquad \text{for all } x \geqq 0.$$

On dividing out the common factor $\prod_{j=1}^{k} (x - \hat{x}_j(n))$ from $H_n(x)$ and $\hat{p}_n(x)$, this implies that

$$x^{\mu_k - k} \leqq |S_n(x)|, \qquad \text{for all } x \geqq 0. \tag{3.25}$$

Thus, for $t \in [0, 1]$, we have

$$\left(1-\frac{1}{n}\right)^{\mu_k - k} \leqq \left(1-\frac{t}{n}\right)^{\mu_k - k} \leqq \left|S_n\left(1-\frac{t}{n}\right)\right|,$$

and so the integral in (3.23) is bounded below by

$$\left(1-\frac{1}{n}\right)^{(\mu_k - k)q} \int_0^1 \prod_{j=1}^{k} |t-t_j(n)|^q \, dt \leqq A', \qquad \text{for all } n \geqq 1.$$

This, in turn, implies that there exists a constant A'' for which

$$\int_0^1 \prod_{j=1}^{k} |t-t_j(n)|^q \, dt \leqq A'', \qquad \text{for all } n \geqq 1. \tag{3.26}$$

By familiar arguments for sequences of polynomials of fixed degree, (3.26) readily implies that the sequence $\left\{ \prod_{j=1}^{k} (t - t_j(n)) \right\}_{n=1}^{\infty}$ is uniformly bounded on any compact set in t of $\mathbb{R}$, and that the $t_j(n)$ are uniformly bounded, as claimed in (3.21).

Now, let $p(t)$ be any limit polynomial derived by taking a suitable subsequence, say $\left\{\prod_{j=1}^{k}(t-t_j(n_i))\right\}_{i=1}^{\infty}$, of $\left\{\prod_{j=1}^{k}(t-t_j(n))\right\}_{n=1}^{\infty}$. For this subsequence we have, from (3.22), that

$$n_i^{nq+1}E_{n_i}^q=\int_0^{n_i}\left(1-\frac{t}{n_i}\right)^{n_i q}\left|S_{n_i}\left(1-\frac{t}{n_i}\right)\right|^q\prod_{j=1}^{k}|t-t_j(n_i)|^q\,dt. \tag{3.27}$$

As the boundeness of the $t_j(n)$ in (3.21) implies that all the $\hat{x}_j(n)$ of (3.20) approach unity as $n\to\infty$, we can apply (3.9) of Lemma 3.2 to deduce that

$$\lim_{i\to\infty}S_{n_i}\left(1-\frac{t}{n_i}\right)=\gamma, \tag{3.28}$$

uniformly on any compact set in t of $\mathbb{R}$. Thus, on taking limits in (3.27), we have

$$\lim_{i\to\infty}n_i^{kq+1}E_{n_i}^q=\gamma^q\int_0^{\infty}e^{-tq}|p(t)|^q\,dt\leqq\gamma^q\,\varepsilon_k^q, \tag{3.29}$$

the last inequality following from (3.19). Therefore, from the definition of ε_k in (3.5) and the uniqueness of the associated extremal polynomial $\tilde{p}(t)$, it follows that $p(t)\equiv\tilde{p}(t)$ and, consequently, that

$$\lim_{i\to\infty}n_i^{kq+1}E_{n_i}^q=\gamma^q\,\varepsilon_k^q.$$

But, as $p(t)$ was *any* limit polynomial, then

$$\lim_{n\to\infty}\prod_{j=1}^{k}(t-t_j(n))=\tilde{p}(t), \tag{3.30}$$

uniformly on any compact set in t of $\mathbb{R}$, and

$$\lim_{n\to\infty}n^{kq+1}E_n^q=\gamma^q\,\varepsilon_k^q. \tag{3.31}$$

Recalling the definition of γ in (3.7), the last equation gives the desired result (3.2) of Theorem 3.1.

Finally, using (3.30), (3.28), and the representation (3.12), the conclusion (3.6) of Theorem 3.1 follows. □

As a consequence of our proof, we mention the following immediate result concerning the zeros of the $\hat{p}_n(x)$ of Theorem 3.1.

Corollary 3.3. *Each of the k positive real zeros of the extremal polynomials $\hat{p}_n(x)$ of (3.4) tend to unity, as $n\to\infty$, with precise rate $O(1/n)$.*

We leave to the reader the statements of some further consequences which generalize to finite q the results of Corollaries 2.2 and 2.3 of [10] for the case $q=\infty$.

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