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ON THE LU FACTORIZATION OF M-MATRICES: CARDINALITY OF THE SET $\mathcal{P}_n^g(A)$

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Abstract. An $n \times n$ M-matrix A is said to admit an LU factorization into $n \times n$ M-matrices if A can be expressed as $A = LU$ where L is an $n \times n$ lower triangular M-matrix and where U is an upper triangular M-matrix. Then, for any given $n \times n$ M-matrix A, let $\mathcal{P}_n^g(A)$ denote the set of all $n \times n$ permutation matrices P such that PAP^T admits an LU factorization into M-matrices with nonsingular L. Our aim here is to determine upper and lower bounds for $|\mathcal{P}_n^g(A)|$, the cardinality of the set $\mathcal{P}_n^g(A)$. This is done in Theorem 4, while in Theorem 2, $|\mathcal{P}_n^g(A)|$ is precisely determined for a special class of $n \times n$ M-matrices.

1. Introduction. If the spectrum, $\sigma(B)$, of an $n \times n$ complex matrix B is defined as

$$\sigma(B) := \{\lambda \in \mathbb{C}: \det[\lambda I - B] = 0\}, \tag{1.1}$$

then an $n \times n$ real matrix $A = [a_{i,j}]$ is said to be an *M-matrix* if

$$a_{i,j} \leqq 0 \quad \text{for all } i \neq j, \quad 1 \leqq i, j \leqq n, \tag{1.2}$$

and if

$$\operatorname{Re} \lambda \geqq 0 \quad \text{for all } \lambda \in \sigma(A). \tag{1.3}$$

It may be somewhat surprising to learn that, despite such a simple definition, the theory and applications of M-matrices form one of the *major* building-blocks of numerical linear algebra (cf. [1] and [6]). Moreover, the applications of M-matrices extend beyond numerical linear algebra to Markov chains, input-output economic models, dynamical systems, mathematical programming, and the compartmental analysis of ecological systems (cf. [1] and [3]).

Such an application as above can give rise to a large sparse system of linear equations whose associated coefficient matrix A is an $n \times n$ M-matrix. For direct methods, comparable to the Gaussian elimination method for solving this system of linear equations, it is of practical interest to know if the associated M-matrix A can be factored as $A = L \cdot U$, where L is an $n \times n$ lower triangular M-matrix, and where U is an upper triangular M-matrix. More precisely, as in [7], an $n \times n$ M-matrix A is said to *admit an LU factorization into n × n M-matrices*, if A can be expressed as

$$A = LU, \tag{1.4}$$

where L is an $n \times n$ lower triangular M-matrix and where U is an $n \times n$ upper triangular M-matrix. As shown in 1962 by Fiedler and Pták [2], any nonsingular M-matrix admits such an LU factorization into M-matrices, with both L and U nonsingular. In 1977, Kuo [5] extended this result by showing that any $n \times n$ irreducible M-matrix (singular or not) admits an LU factorization (1.4) into M-matrices with, say, L nonsingular. For the remaining set of M-matrices, it is easy to see that not every singular and reducible $n \times n$ M-matrix admits an LU factorization into

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M-matrices with L nonsingular, as the particular matrix

$$A_1 = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \tag{1.5}$$

directly shows. However, if $\mathscr{P}_n$ denotes the collection of all $n \times n$ permutation matrices, Kuo [5] has shown that, for any $n \times n$ M-matrix A, the subset of $\mathscr{P}_n$, defined by

$$\begin{aligned}\mathscr{P}_n^g(A) := \{P \in \mathscr{P}_n : PAP^T &\text{ admits an } LU \text{ factorization into} \\ &M\text{-matrices with nonsingular } L\},\end{aligned} \tag{1.6}$$

is *never* empty. (Here, the superscript "g" in (1.6) refers to "good" permutations.) If $|\mathscr{P}_n^g(A)|$ denotes the cardinality of $\mathscr{P}_n^g(A)$ (i.e., the exact number of its elements), then the fact that $\mathscr{P}_n^g(A)$ is not empty implies (since $\mathscr{P}_n$ contains $n!$ elements) that

$$1 \leqq |\mathscr{P}_n^g(A)| \leqq n!, \tag{1.7}$$

for every $n \times n$ M-matrix A. From the above results, we remark that equality must evidently hold on the right in (1.7) for any nonsingular or irreducible $n \times n$ M-matrix A. From Funderlic and Plemmons [3], the same is true in (1.7) for any symmetric M-matrix A and for any M-matrix A for whicy $y^TA \geqq 0$ for some $y > 0$. Later (cf. (2.15)), we shall see that the first inequality in (1.7) is *sharp* for every $n \geqq 1$.

Our aim in this note is to determine upper and lower bounds for $|\mathscr{P}_n^g(A)|$, for any $n \times n$ M-matrix A. The outline of this note is as follows. We conclude this section with some needed notation, and in § 2, after giving some definitions, state our main results and give some applications of these results. The proofs of our main results are then given in § 3.

We assume, without loss of generality, that the $n \times n$ M-matrix A is in *normal reduced form* (cf. [7]), i.e.,

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,l} \\ & A_{2,2} & \cdots & A_{2,l} \\ & & \ddots & \vdots \\ 0 & & & A_{l,l} \end{bmatrix}, \tag{1.8}$$

where each diagonal submatrix $A_{j,j}$ is irreducible $(1 \leqq j \leqq l)$. (As in [7], it is convenient to define all 1×1 null matrices here to be irreducible.) Of course, if A is irreducible, then $l = 1$ in (1.8). For large matrices, we remark that good software exists, for permuting the rows and columns of A to bring A into the form (1.8). For this, see George and Gustavson [4].

Next, if we define R_A as follows

$$R_A := \{j \text{ with } 1 \leqq j \leqq l : A_{j,j} \text{ is a singular and irreducible } M\text{-matrix}\}, \tag{1.9}$$

then $R_A \neq \emptyset$ if and only if A is a singular M-matrix. Continuing, we define the $l \times l$ upper triangular matrix $\mathscr{B}_A := [b_{i,j}]$, derived from A in (1.8) by means of

$$b_{i,j} := \begin{cases} 1 & \text{if } i \neq j \text{ and if } A_{i,j} \not\equiv \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.10}$$

Its directed graph $G_l(\mathscr{B}_A)$ on l vertices $V_1, V_2, \cdots, V_l$, is called the *block-directed graph* for the matrix A of (1.8). (As in [6] or [7], a *path* in $G_l(\mathscr{B}_A)$ from vertex V_i to vertex V_s is a sequence $\{b_{k_r,k_{r+1}}\}_{r=1}^{j}$ with $j \geqq 1$, $b_{k_r,k_{r+1}} \neq 0$, and with $k_1 = i$ and $k_{j+1} = s$.) For additional notation, with $\langle m \rangle := \{1, 2, \cdots, m\}$, let $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ be a nonempty subset of $\langle n\rangle$, and let $A[\alpha]$ denote the induced principal submatrix of the $n\times n$ matrix $A=[a_{i,j}]$, determined by α, i.e.,

$$A[\alpha]=[a_{i,j}], \quad \text{where } i, j\in\alpha. \tag{1.11}$$

As in [7], we shall say that α is a *proper* subset of $\langle n\rangle$ if $\emptyset\neq\alpha\subsetneqq\langle n\rangle$.

If R_A of (1.9) is nonempty, then for each $j\in R_A$, we define the set

$$S_j := \{k\neq j\text{: there is a path in } G_l(\mathscr{B}_A) \text{ from vertex } V_k \text{ to vertex } V_j\}. \tag{1.12}$$

Because of the triangular form of (1.8), we note that S_j can contain only integers k satisfying $1\leqq k<j$, so that S_1, for example, is empty by definition. It is also convenient to say that

$$S_j \text{ is } \textit{full} \quad \text{iff} \quad S_j=\langle j-1\rangle. \tag{1.13}$$

2. Main results and applications. To state our first result, let A be an $n\times n$ singular M-matrix in normal reduced form (1.8) so that $R_A\neq\emptyset$, and suppose that S_j is not empty for some $j\in R_A$. Then, set

$$\mu := \max\{j\in R_A\colon S_j\neq\emptyset\}, \tag{2.1}$$

and *assume that* S_μ *is full.* Note, from (1.13), that the assumption that S_μ is full implies that $\mu>1$. With this value of μ, we define the following two principal submatrices of A, which are evidently M-matrices:

$$B := \begin{bmatrix} A_{1,1} & \cdots & A_{1,\mu} \\ & \ddots & \vdots \\ 0 & & A_{\mu,\mu}\end{bmatrix}, \qquad C := \begin{bmatrix} A_{1,1} & \cdots & A_{1,\mu-1} \\ & \ddots & \vdots \\ 0 & & A_{\mu-1,\mu-1}\end{bmatrix}. \tag{2.2}$$

This brings us to the statement of our first result, whose proof will be given in § 3.

THEOREM 1. (reduction algorithm). *Let A be an $n\times n$ M-matrix in normal reduced form* (1.8). *If $R_A=\emptyset$, or if $R_A\neq\emptyset$ and if $S_j=\emptyset$ for each $j\in R_A$* (cf. (1.12)), *then*

$$|\mathscr{P}_n^g(A)|=n!. \tag{2.3}$$

Otherwise, let μ be defined as in (2.1), *and assume that S_μ is full. If the matrix C of* (2.2) *is $s\times s$ and if $A_{\mu,\mu}$ is $m\times m$, then*

$$|\mathscr{P}_n^g(A)|=\frac{n!\cdot m|\mathscr{P}_s^g(C)|}{s!(s+m)}. \tag{2.4}$$

We remark that since the order of the matrix C of (2.2) is necessarily *less* than that of A, we can view Theorem 1 as a *reduction algorithm which* precisely relates $|\mathscr{P}_n^g(A)|$ for A to $|\mathscr{P}_s^g(C)|$ for the smaller matrix C. Of course, if C is nonsingular (so that its associated set R_C of (1.9) is empty), or if C is singular and its associated sets S_j of (1.12) are empty for all $j\in R_C$ (as is the case when C is irreducible), then $|\mathscr{P}_s^g(C)|=s!$, and the reduction algorithm necessarily *terminates*. Otherwise, the reduction algorithm can be continued if C satisfies the hypotheses of Theorem 1. Assuming that $R_A\neq\emptyset$, a sufficient condition that this reduction algorithm can be continued to termination is that

$$\text{for every } j\in R_A, \text{ either } S_j=\emptyset \text{ or } S_j=\langle j-1\rangle. \tag{2.5}$$

Now, if $R_A\neq\emptyset$, we further set

$$R_A^F := \{j\in R_A\colon S_j=\langle j-1\rangle\}, \tag{2.6}$$

and if $R_A^F \neq \varnothing$, we list (for convenience) its elements in decreasing order, i.e.,

$$R_A^F := \{\mu_1, \mu_2, \cdots, \mu_k\}, \quad \text{where } n \geqq \mu_1 > \mu_2 > \cdots > \mu_k \geqq 2. \tag{2.7}$$

For each $\mu_j \in R_A^F$, let t_j be defined as the order of the matrix B of (2.2) with $\mu = \mu_j$, and let s_j be similarly defined as the order of the matrix C of (2.2) with $\mu = \mu_j$. If m_j is the order of the matrix A_{μ_j,μ_j}, we note that $t_j = s_j + m_j$, $j = 1, 2, \cdots, k$.

With this notation, we state our next result, an extension of Theorem 1, whose proof will also be given in § 3.

THEOREM 2. *Let A be an $n \times n$ M-matrix, in normal reduced form* (1.8) *which satisfies* (2.5) *if $R_A \neq \varnothing$. If $R_A = \varnothing$, or if $R_A \neq \varnothing$ with $R_A^F = \varnothing$* (cf. (2.6)), *then*

$$|\mathcal{P}_n^g(A)| = n!. \tag{2.8}$$

Otherwise, with the above notation,

$$|\mathcal{P}_n^g(A)| = n! \frac{\prod_{j=1}^k m_j}{\prod_{j=1}^k t_j}, \tag{2.9}$$

where $R_A^F = \{\mu_1, \mu_2, \cdots, \mu_k\}$.

We next consider applications of Theorem 2. As our first application, consider the $n \times n$ singular reducible M-matrix (in normal reduced form)

$$D = \left[\begin{array}{c|c} D_{1,1} & D_{1,2} \\ \hline 0 & D_{2,2} \end{array}\right], \tag{2.10}$$

where $D_{1,1}$ is an $n_1 \times n_1$ nonsingular irreducible M-matrix, where $D_{2,2}$ is an $n_2 \times n_2$ singular irreducible M-matrix (with $n_1 + n_2 = n$), and where $D_{1,2} \not\equiv \mathcal{O}$. In this example, $R_D = \{2\}$ and $S_2 = \{1\}$, so that the hypotheses of (2.5) are satisfied. Thus, $R_D^F = \{2\}$, so that $\mu_1 = 2$, $t_1 = n$, and $m_1 = n_2$. Applying (2.9) of Theorem 2 gives

$$|\mathcal{P}_n^g(D)| = n_2 \cdot (n-1)!, \tag{2.11}$$

the result of [7, Thm. 4]. Thus, Theorem 2 here generalizes [7, Thm. 4].

As a second application of Theorem 2, consider the following 12×12 singular reducible M-matrix (in normal reduced form)

$$E = \left[\begin{array}{c|c|c|c|c|c}
\begin{matrix}2 & -1\\-1 & 2\end{matrix} & \begin{matrix}0 & -1\\-1 & 0\end{matrix} & 0 & \begin{matrix}0 & -1\\-1 & 0\end{matrix} & 0 & \begin{matrix}0 & -1\\-1 & 0\end{matrix} \\ \hline
0 & \begin{matrix}1 & -1\\-1 & 1\end{matrix} & 0 & \begin{matrix}0 & -1\\-1 & 0\end{matrix} & 0 & \begin{matrix}0 & -1\\-1 & 0\end{matrix} \\ \hline
0 & 0 & \begin{matrix}2 & -1\\-1 & 2\end{matrix} & \begin{matrix}0 & -1\\-1 & 0\end{matrix} & 0 & \begin{matrix}0 & -1\\-1 & 0\end{matrix} \\ \hline
0 & 0 & 0 & \begin{matrix}1 & -1\\-1 & 1\end{matrix} & 0 & \begin{matrix}0 & -1\\-1 & 0\end{matrix} \\ \hline
0 & 0 & 0 & 0 & \begin{matrix}1 & -1\\-1 & 1\end{matrix} & \begin{matrix}0 & -1\\-1 & 0\end{matrix} \\ \hline
0 & 0 & 0 & 0 & 0 & \begin{matrix}2 & -1\\-1 & 2\end{matrix}
\end{array}\right]. \tag{2.12}$$

In this case, (2.5) is satisfied, and $R_E = \{2, 4, 5\}$, and $R_E^F = \{2, 4\}$, so that $\mu_1 = 4$, $\mu_2 = 2$, $t_1 = 8$, $t_2 = 4$, and $m_2 = m_4 = 2$. Applying (2.9) yields

$$|\mathcal{P}_{12}^{g}(E)| = \frac{12!}{8}. \tag{2.13}$$

As our final application of Theorem 2, consider the particular upper triangular $n \times n$ singular M-matrix, defined by

$$H_n := \begin{bmatrix} 0 & -1 & -1 & \cdots & -1 & -1 \\ & 0 & -1 & \cdots & -1 & -1 \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & -1 & -1 \\ & & & & 0 & -1 \\ 0 & & & & & 0 \end{bmatrix}. \tag{2.14}$$

In this case, (2.5) is again satisfied, and $R_{H_n} = \{1, 2, \cdots, n\}$, $R_{H_n}^F = \{n, n-1, \cdots, 2\}$, and $m_j = 1$ for each $1 \leqq j \leqq n-1$. Applying (2.9) of Theorem 2 gives that

$$|\mathcal{P}_{n}^{g}(H_n)| = 1. \tag{2.15}$$

This example constructively shows that the first inequality in (1.7) is *sharp* for every $n \geqq 1$.

While Theorem 2 precisely determines $|\mathcal{P}_n^g(A)|$ for those $n \times n$ singular reducible M-matrices A satisfying (2.5), we next seek upper and lower bounds for $|\mathcal{P}_n^g(A)|$ for singular reducible M-matrices which do *not* satisfy (2.5).

Consider two $n \times n$ M-matrices A and B which are in normal reduced form

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,l} \\ & A_{2,2} & \cdots & A_{2,l} \\ & & \ddots & \vdots \\ 0 & & & A_{l,l} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,m} \\ & B_{2,2} & \cdots & B_{2,m} \\ & & \ddots & \vdots \\ 0 & & & B_{m,m} \end{bmatrix}, \tag{2.16}$$

where the diagonal submatrices $A_{j,j}$ and $B_{j,j}$ are irreducible. We say that A and B are *graph-compatible* if (cf. (2.16))

(2.17)
- (i) $l = m$;
- (ii) the order of $A_{j,j}$ is equal to the order of $B_{j,j}$ for each j with $1 \leqq j \leqq l$;
- (iii) $R_A = R_B$ (cf. (1.9)).

With $S_j(A)$ and $S_j(B)$ denoting the sets of (1.12) associated with A and B when $R_A \neq \emptyset \neq R_B$, we come to:

PROPOSITION 3. *Let A and B be two $n \times n$ M-matrices which are graph-compatible* (cf. (2.17)). *If $R_A \neq \emptyset$ and if*

$$S_j(A) \subseteq S_j(B) \text{ for each } j \in R_A, \tag{2.18}$$

then

$$|\mathcal{P}_n^g(A)| \geqq |\mathcal{P}_n^g(B)|, \tag{2.19}$$

with strict inequality holding in (2.19) *if $S_j(A) \subsetneqq S_j(B)$ for some $j \in R_A$.*

We now use Proposition 3 as follows. Consider any $n \times n$ M-matrix A which is in normal reduced form (1.8). We shall construct two $n \times n$ singular reducible

M-matrices, $\underset{\sim}{A}$ and $\bar{A}$, which are graph-compatible with A. Specifically, with

$$(2.20)\qquad \underset{\sim}{A} := \begin{bmatrix} \underset{\sim}{A}_{1,1} & \underset{\sim}{A}_{1,2} & \cdots & \underset{\sim}{A}_{1,l} \\ & \underset{\sim}{A}_{2,2} & \cdots & \underset{\sim}{A}_{2,l} \\ & & \ddots & \vdots \\ 0 & & & \underset{\sim}{A}_{l,l} \end{bmatrix}, \qquad \bar{A} := \begin{bmatrix} \bar{A}_{1,1} & \bar{A}_{1,2} & \cdots & \bar{A}_{1,l} \\ & \bar{A}_{2,2} & \cdots & \bar{A}_{2,l} \\ & & \ddots & \vdots \\ 0 & & & \bar{A}_{l,l} \end{bmatrix},$$

we set

$$(2.21)\qquad \begin{aligned} &\underset{\sim}{A}_{i,i} = \bar{A}_{i,i} = A_{i,i} && \text{for each } 1 \leqq i \leqq l; \\ &\underset{\sim}{A}_{i,j} = \bar{A}_{i,j} = A_{i,j} && \text{for each } j \notin R_A, \text{ and all } 1 \leqq i \leqq l; \\ &\underset{\sim}{A}_{i,j} = \bar{A}_{i,j} = A_{i,j} && \text{for each } j \in R_A \text{ such that either } S_j(A) = \varnothing \\ & && \text{or } S_j(A) = \langle j-1 \rangle, \text{ and all } 1 \leqq i \leqq l. \end{aligned}$$

Of course, if $R_A = \varnothing$ or if $R_A \neq \varnothing$ and A satisfies (2.5), then $\underset{\sim}{A}$ and $\bar{A}$ are fully defined, with $\underset{\sim}{A} = A = \bar{A}$. Otherwise, suppose there is a $j \in R_A$ for which $\varnothing \neq S_j(A) \subsetneqq \langle j-1 \rangle$. For such j's, we change zero blocks $A_{i,j}$ of A to nonzero blocks $\underset{\sim}{A}_{i,j}$ in the upper triangular part of the jth column of A in such a way that

$$(2.22)\qquad S_j(\underset{\sim}{A}) = \langle j-1 \rangle \quad \text{for those } j \in R_A \text{ for which } \varnothing \neq S_j(A) \subsetneqq \langle j-1 \rangle.$$

Similarly, we change all nonzero blocks $A_{i,j}$, in the upper triangular part of the jth column of A, to be identically zero, thereby defining $\bar{A}_{i,j} \equiv \mathcal{O}$ for all $1 \leqq i < j$, so that

$$(2.23)\qquad S_j(\bar{A}) = \varnothing \quad \text{for those } j \in R_A \text{ for which } \varnothing \neq S_j(A) \subsetneqq \langle j-1 \rangle.$$

Clearly, the matrices $\underset{\sim}{A}$ and $\bar{A}$ are, by construction, M-matrices which are both graph-compatible with A. Moreover, if $R_A \neq \varnothing$, the matrices $\underset{\sim}{A}$ and $\bar{A}$ are such that (2.5) is satisfied for each of these matrices, and also such that

$$(2.24)\qquad S_j(\bar{A}) \subseteq S_j(A) \subseteq S_j(\underset{\sim}{A}) \quad \text{for each } j \in R_A.$$

Now, $|\mathcal{P}_n^g(\bar{A})|$ and $|\mathcal{P}_n^g(\underset{\sim}{A})|$ can be exactly computed from Theorem 2, so that from (2.19) of Proposition 3, we immediately have

THEOREM 4. *Let A be an $n \times n$ M-matrix in normal reduced form* (1.8). *With the $n \times n$ M-matrices $\underset{\sim}{A}$ and $\bar{A}$ of* (2.21)–(2.23), *then either $R_A = \varnothing$ or $R_A \neq \varnothing$ and A satisfies* (2.5), *so that $\underset{\sim}{A} = A = \bar{A}$ and*

$$(2.25)\qquad |\mathcal{P}_n^g(\underset{\sim}{A})| = |\mathcal{P}_n^g(A)| = |\mathcal{P}_n^g(\bar{A})|,$$

or $R_A \neq \varnothing$ and A does not satisfy (2.5), *so that*

$$(2.26)\qquad |\mathcal{P}_n^g(\underset{\sim}{A})| < |\mathcal{P}_n^g(A)| < |\mathcal{P}_n^g(\bar{A})|,$$

where $|\mathcal{P}_n^g(\underset{\sim}{A})|$ and $|\mathcal{P}_n^g(\bar{A})|$ can be exactly determined from (2.9) *of Theorem* 2.

As an illustration of Theorem 4, consider the particular singular reducible M-matrix

$$(2.27)\qquad J = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For this matrix, its associated graph-compatible matrices $\underset{\sim}{J}$ and $\bar{J}$ can be taken to be

$$(2.28)\qquad \underset{\sim}{J}=\begin{bmatrix} 1 & -1 & -1 & -1\\ 0 & 0 & -1 & -1\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 1\end{bmatrix}\quad\text{and}\quad \bar{J}=\begin{bmatrix} 1 & -1 & 0 & -1\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

Here, the inequality of (2.26) of Theorem 4 can be computed to give

$$(2.29)\qquad 4<|\mathcal{P}_4^g(J)|<12.$$

By direct computation, we find, on the other hand, that $|\mathcal{P}_4^g(J)|=8$.

As a final remark, suppose that an $n\times n$ M-matrix A is the direct sum of k M-matrices, i.e., in block-diagonal form,

$$(2.30)\qquad A=\operatorname{diag}[A_{1,1},A_{2,2},\cdots,A_{k,k}],$$

where each $A_{j,j}$ is an $m_j\times m_j$ M-matrix. It is easy to see that

$$(2.31)\qquad |\mathcal{P}_n^g(A)|=\frac{n!\prod_{i=1}^{k}|\mathcal{P}_{m_i}^g(A_{i,i})|}{\prod_{i=1}^{k}(m_i!)},\quad\text{where } n=\sum_{i=1}^{k} m_i.$$

The point of this remark is that if (2.30) is valid, then Theorems 2 and 4 should be applied *only* to the matrices $A_{i,i}$, $1\leqq i\leqq k$.

To illustrate this last remark, consider the following matrix

$$(2.32)\qquad A=\left[\begin{array}{cc:cc} \square & \times & & \\ \hdashline & \Theta & & \\ \hline & & \Theta & \times\\ \hdashline & & & \Theta\end{array}\right]=\left[\begin{array}{c|c} A_{1,1} & \mathcal{O}\\ \hline \mathcal{O} & A_{2,2}\end{array}\right],$$

where $\square$, Θ denote respectively nonsingular and singular irreducible M-matrices, where blank blocks are identically zero, and where $\times$'s denote nonzero blocks. In this example, $R_A=\{2,3,4\}$, $S_2=\{1\}$, $S_3=\varnothing$, and $S_4=\{3\}$; moreover, as $R_A\neq\varnothing$ and as (2.5) is not satisfied by A, Theorem 2 does not apply to A. However, A of (2.32) is the direct sum of the two matrices $A_{1,1}$ and $A_{2,2}$. As Theorem 2 *can* be applied to $A_{1,1}$ and $A_{2,2}$, then (2.31) can be applied, and $|\mathcal{P}_n^g(A)|$ can be precisely determined.

3. Proofs of results. For the convenience of the reader, we state below two results from [7] which will be used below.

THEOREM A. *Let A be an $n\times n$ M-matrix. Then, the following are equivalent*:

(i) *A admits an LU factorization into M-matrices with nonsingular L*;

(ii) *for every proper subset $\alpha=\{\alpha_1,\alpha_2,\cdots,\alpha_k\}$ of $\langle n\rangle$ for which $A[\alpha]$ is singular and irreducible, there is no path in the directed graph $G_n(A)$ of A from vertex v_t to vertex v_{α_j} for any $t>\alpha_k$ and any $1\leqq j\leqq k$.*

THEOREM B. *Let $A=[a_{i,j}]$ be an $n\times n$ M-matrix. Then, the following are equivalent*:

(i) *there exists an $x>0$ such that $x^TA\geqq 0$*;

(ii) $|\mathcal{P}_n^g(A)|=n!$;

(iii) *for every proper subset $\alpha=\{\alpha_1,\alpha_2,\cdots,\alpha_k\}$ of $\langle n\rangle$ for which $A[\alpha]$ is singular and irreducible, then $a_{t,p}=0$ for all $t\notin\alpha$ and all $p\in\alpha$.*

Proof of Theorem 1. As (2.3) is immediate if $R_A=\varnothing$, assume first that $R_A\neq\varnothing$ and that the set S_j of (1.12) is empty for every $j\in R_A$. This implies that, for each $j\in R_A$, $A_{i,j}\equiv\mathcal{O}$ for each $1\leqq i\leqq l$ with $i\neq j$. Since each singular and irreducible submatrix of A of (1.8) must be some diagonal submatrix $A_{j,j}$ of A with $j \in R_A$, it follows that (iii) of Theorem B above is valid; whence, from (ii) of Theorem B, $|\mathcal{P}_n^g(A)| = n!$. This gives (2.3).

Next, assume that $S_j \neq \varnothing$ for some $j \in R_A$. With μ defined as in (1.13), we further assume that S_μ is full. The idea of the proof now is to use the equivalence of (i) and (ii) in Theorem A, in two stages, to deduce the desired result (2.4) of Theorem 1. We remark that if the directed graph $G_n(A)$ of the $n \times n$ M-matrix $A = [a_{i,j}]$ is associated with the n vertices $v_1, v_2, \cdots, v_n$, if σ is any permutation (1-1 transformation) on $\langle n \rangle$, and if its associated permutation matrix P_σ is defined by $P_\sigma = [\delta_{i,\sigma(j)}]$, then the directed graph $G_n(P_\sigma A P_\sigma^T)$ for $P_\sigma A P_\sigma^T$ is simply obtained by relabeling the vertices of $G_n(A)$ from v_j to $v_{\sigma(j)}$, while keeping all arcs intact. This observation will allow us to determine *which* rearrangements (permutations) of $\langle n \rangle$ are such that (ii) of Theorem A, applied to these rearrangements, is valid.

If the matrix B of (2.2) is $t \times t$, we first wish to establish that

$$|\mathcal{P}_n^g(A)| = |\mathcal{P}_t^g(B)| \cdot \frac{n!}{t!}. \tag{3.1}$$

Of course, if $t = n$, then $A = B$ and (3.1) trivially holds. Thus, we may assume that $t < n$. Consider any rearrangements of the first t positive integers, say $\{\nu_1, \nu_2, \cdots, \nu_t\}$, and consider any rearrangement $\{\tau_1, \tau_2, \cdots, \tau_{n-t}\}$ of the remaining positive integers $\{t+1, t+2, \cdots, n\}$. We then intersperse the integers of $\{\tau_1, \cdots, \tau_{n-t}\}$ among the integers of $\{\nu_1, \cdots, \nu_t\}$, thereby forming $\{\omega_1, \omega_2, \cdots, \omega_n\}$, a rearrangement of the first n integers, in such a way that $\{\omega_1, \cdots, \omega_n\} \backslash \{\tau_1, \cdots, \tau_{n-t}\} = \{\nu_1, \cdots, \nu_t\}$ and such that $\{\omega_1, \cdots, \omega_n\} \backslash \{\nu_1, \cdots, \nu_t\} = \{\tau_1, \cdots, \tau_{n-t}\}$. We claim that the number of ways of interspersing $\{\tau_1, \cdots, \tau_{n-t}\}$ with $\{\nu_1, \cdots, \nu_t\}$ is

$$K = \frac{n!}{t!}. \tag{3.2}$$

To see this, each distinct method of interspersing $\{\tau_1, \cdots, \tau_{n-t}\}$ among the integers of $\{\nu_1, \cdots, \nu_t\}$ applies equally well to each rearrangement $\{\nu_1', \cdots, \nu_t'\}$ of the first t positive integers. Thus, there are exactly the *same* number, say K, of ways of interspersing the integers of $\{\tau_1, \cdots, \tau_{n-t}\}$ among the integers of *each* rearrangement of the first t integers. Clearly, the totality of arrangements of $\{\omega_1, \cdots, \omega_n\}$ which can be obtained is, on one hand, $K \cdot t!$, while on the other hand, it is necessarily $n!$, which gives (3.2).

Next, we make the observation that if the rearrangement $\{\nu_1, \cdots, \nu_t\}$ corresponds to an element of $\mathcal{P}_t^b(B) := \langle n \rangle \backslash \mathcal{P}_t^g(B)$, then it is easily seen that every interspersing of the integers of any rearrangement $\{\tau_1, \cdots, \tau_{n-t}\}$ by definition corresponds to an element of $\mathcal{P}_n^b(A)$. Thus, to obtain a rearrangement $\{\omega_1, \cdots, \omega_n\}$ in $\mathcal{P}_n^g(A)$, it is necessary to *begin* with a rearrangement $\{\nu_1, \cdots, \nu_t\}$ which is in $\mathcal{P}_t^g(B)$, followed by *any* interspersing of *any* $\{\tau_1, \cdots, \tau_{n-t}\}$. (The reason that this is valid is that the $n-t$ integers $\{\tau_1, \cdots, \tau_{n-t}\}$ necessarily correspond to vertices in the directed graph $G_n(A)$ of A which, by construction, have *no* path to the singular irreducible submatrix $A_{\mu,\mu}$, and hence play no role in applying (ii) of Theorem A to $A_{\mu,\mu}$.) Thus, using (3.2), $|\mathcal{P}_n^g(A)|$ is given by

$$|\mathcal{P}_n^g(A)| = \frac{n!}{t!} |\mathcal{P}_t^g(B)|. \tag{3.3}$$

We now relate, in the second part of the proof, the quantities $|\mathcal{P}_t^g(B)|$ and $|\mathcal{P}_s^g(C)|$, where B and C are defined in (2.2). Since S_μ is full by hypothesis, then $S_\mu = \langle \mu - 1 \rangle$.

By definition, the matrices $A_{\mu,\mu}$, C and B (cf. (2.2)) are respectively of orders m, s and t, with $t=m+s$. In analogy to the first part of the proof, we consider any rearrangement $\{\tau_1,\cdots,\tau_s\}$ of the first s positive integers, and any rearrangement $\{\eta_1,\cdots,\eta_m\}$ of the integers $\{s+1,\cdots,t\}$, and we intersperse $\{\eta_1,\cdots,\eta_m\}$ among the integers of $\{\tau_1,\cdots,\tau_s\}$, thereby forming $\{\omega_1,\omega_2,\cdots,\omega_t\}$. As before, to obtain an element in $\mathcal{P}_t^g(B)$, it is necessary to *begin* with a rearrangement $\{\tau_1,\cdots,\tau_s\}$ which is in $\mathcal{P}_s^g(C)$. Moreover, because by hypothesis $S_\mu=\langle\mu-1\rangle$, we see from (ii) of Theorem A that the final element ω_t of $\{\omega_1,\cdots,\omega_t\}$ in $\mathcal{P}_t^g(B)$ *must* be from $\{\eta_1,\cdots,\eta_m\}$. For each fixed $\{\tau_1,\cdots,\tau_s\}$ in $\mathcal{P}_s^g(C)$, it is easily seen that there are the *same* number, namely $m\cdot(t-1)!/s!$ of such interspersings of $\{\eta_1,\cdots,\eta_m\}$, such that the last element ω_t is from $\{\eta_1,\cdots,\eta_m\}$. Thus,

$$|\mathcal{P}_t^g(B)|=\frac{|\mathcal{P}_s^g(C)|\cdot m\cdot(t-1)!}{s!}. \tag{3.4}$$

If we combine (3.4) with (3.3), we obtain (since $t=s+m$) the desired result (2.4). $\square$

Proof of Theorem 2. As (2.8) is immediate if $R_A=\varnothing$, assume first that $R_A\neq\varnothing$ and that $R_A^F=\varnothing$. But, $R_A^F=\varnothing$ implies from (2.5) that $S_j=\varnothing$ for each $j\in R_A$, which with (2.3) of Theorem 1 gives that $|\mathcal{P}_n^g(A)|=n!$ in (2.8). Hence, we may assume that R_A^F is not empty, so that from (2.7), $R_A^F=\{\mu_1,\mu_2,\cdots,\mu_k\}$ where $n\geqq\mu_1>\mu_2>\cdots>\mu_k\geqq 2$. Now, let $C^{(j)}$ denote the matrix C of (2.2) when $\mu=\mu_j$, $j=1,2,\cdots,k$. In addition, we set

$$C^{(0)}:=A. \tag{3.5}$$

From the discussion preceding Theorem 2, s_j denotes the order of each $C^{(j)}$, so that $s_0:=n$. Similarly, m_j denotes the order of the matrix A_{μ_j,μ_j}. Then, applying (2.4) of Theorem 1 to $C^{(j)}$ yields

$$|\mathcal{P}_{s_j}^g(C^{(j)})|=\frac{(s_j)!m_{j+1}|\mathcal{P}_{s_{j+1}}^g(C^{(j+1)})|}{(s_{j+1})!(t_{j+1})},\qquad j=0,1,\cdots,k-1, \tag{3.6}$$

where $t_j:=s_j+m_j$. On multiplying the quantities of (3.6) for all $j=0,1,\cdots,k-1$, we obtain (since $s_0:=n$)

$$|\mathcal{P}_n^g(A)|=\frac{n!\prod_{j=1}^k m_j|\mathcal{P}_{s_k}^g(C^{(k)})|}{(s_k)!\cdot\prod_{j=1}^k t_j}. \tag{3.7}$$

But, since the irreducible diagonal submatrices of $C^{(k)}$ are either nonsingular, or singular with associated sets S_j empty, from (2.5) and (2.7), $|\mathcal{P}_{s_k}^g(C^{(k)})|=(s_k)!$, and (3.7) then reduces to the desired result (2.9). $\square$

Proof of Proposition 3. With the hypotheses of Proposition 3, consider any permutation P in $\mathcal{P}_n^g(B)$. From the equivalence of (i) and (ii) in Theorem A, it is easy to verify that the hypothesis of graph-compatibility and the inclusions of (2.18) imply that P is also in $\mathcal{P}_n^g(A)$, whence $|\mathcal{P}_n^g(A)|\geqq|\mathcal{P}_n^g(B)|$, the desired inequality of (2.19).

Next, suppose that there is a $j\in R_A$ for which $S_j(A)\subsetneqq S_j(B)$, along with $S_k(A)\subseteq S_k(B)$ for each $k\in R_A$, and let

$$s:=\max\{k\colon k\in S_j(B)\backslash S_j(A)\},\quad \text{where } s<j. \tag{3.8}$$

By definition, there is a path from vertex V_s to vertex V_j in the block-directed graph for the matrix B, but no such path in the block-directed graph for the matrix A. With A in reduced normal form (2.16) and with $\alpha:=\{s,s+1,\cdots,j\}$, set

$$T_s^A:=\{t\colon s\leqq t\leqq j \text{ and there is a path from } V_s \text{ to } V_j \text{ for } A\}\cup\{s\}. \tag{3.9}$$

By definition,

$$s \in T_s^A \quad \text{and} \quad j \in \{\alpha \setminus T_s^A\}. \tag{3.10}$$

Then, as in the proof of Theorem 1, with each vertex V_i, we associate n_i positive integers (where $A_{i,i}$ and $B_{i,i}$ are $n_i \times n_i$ integers), numbered consecutively so that the integers associated with V_1 are $\{1, 2, \cdots, n_1\}$, those associated with V_2 are $\{n_1 + 1, \cdots, n_1 + n_2\}$, etc. Now, alter $\langle n \rangle$ (thereby forming a rearrangement of $\langle n \rangle$) by simply removing the consecutive integers, corresponding in sequence to the vertices V_l in T_s^A, and placing them (without changing their relative positions) immediately after the last integer associated with the last vertex of V_j. Using (3.10), Theorem A, and the fact that there *is* a path from vertex V_s to vertex V_j for the matrix B, this new rearrangement of $\langle n \rangle$ can be seen to be in the set $\mathcal{P}_n^g(A)$, but not in the set $\mathcal{P}_n^g(B)$. Thus, $|\mathcal{P}_n^g(A)| > |\mathcal{P}_n^g(B)|$, which gives strict inequality in (2.19). □

Proof of Theorem 4. The proof of Theorem 4 follows easily from Proposition 3. First, if $R_A = \emptyset$ or if $R_A = \emptyset$ and A satisfies (2.5), the construction of $\underset{\sim}{A}$ and $\bar{A}$ is such that $\underset{\sim}{A} = A = \bar{A}$ in this case, from which (2.25) follows. Otherwise, assume $R_A \neq \emptyset$ and that A does not satisfy (2.5). Hence, there exists a $j \in R_A$ for which $\emptyset \neq S_j(A) \subsetneqq \langle j-1 \rangle$. For this j, the construction of $\underset{\sim}{A}$ and $\bar{A}$ from (2.22)–(2.23) shows that

$$S_j(\bar{A}) = \emptyset \neq S_j(A) \subsetneqq S_j(\underset{\sim}{A}) = \langle j-1 \rangle, \tag{3.11}$$

as well as

$$S_k(\bar{A}) \subseteq S_k(A) \subseteq S_k(\underset{\sim}{A}) \quad \text{for which } k \in R_A. \tag{3.12}$$

Thus, strict inequality holds in (2.19) of Proposition 3, i.e.,

$$|\mathcal{P}_n^g(\underset{\sim}{A})| < |\mathcal{P}_n^g(A)| < |\mathcal{P}_n^g(\bar{A})|, \tag{3.13}$$

which gives the desired result (2.26) of Theorem 4. □

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