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On the Overconvergence of Complex Interpolating Polynomials

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1. INTRODUCTION

Recently, there has been a good deal of interest in extensions of a beautiful and well-known result of J. L. Walsh [4, p. 153] on the overconvergence of differences of interpolating polynomials. As background for Walsh's result, let $\rho > 1$ be a fixed real number, and let

$$A_\rho := \{f(z): f \text{ is analytic in } |z| < \rho \text{ and has a singularity on } |z| = \rho\}. \quad (1.1)$$

Further, let $Z = \{z_{k,n}\}$ be an infinite triangular interpolation matrix whose entries satisfy

$$1 \leqslant |z_{k,n}| < \rho \qquad (k = 1, 2,..., n; n = 1, 2,...). \quad (1.2)$$

Then, for any $f \in A_\rho$, let $p_{n-1}(z, Z, f)$ denote the unique polynomial (of degree at most $n-1$) which interpolates f in the n points $\{z_{k,n}\}_{k=1}^n$ of the nth row of Z, i.e.,

$$p_{n-1}(z_{k,n}, Z, f) = f(z_{k,n}), \qquad k = 1, 2,..., n; n = 1, 2,.... \quad (1.3)$$

We do *not* assume that the entries $\{z_{k,n}\}_{k=1}^n$ in the nth row of Z are distinct. In the case of repeated points in the nth row of Z, the interpolation in (1.3)

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will be understood to be in the Hermite (derivative) sense. When the entries in each row of Z are just the nth roots of unity, i.e., when

$$z_{k,n} := \exp\{2\pi ki/n\}, \qquad k = 1, 2,..., n;\ n = 1, 2,...,$$

the associated triangular interpolation matrix will be denoted by E. Similarly, O denotes the triangular interpolation matrix all of whose entries are zero.

Next, if $f(z)$ in A_ρ has the expansion $f(z) = \sum_{j=0}^{\infty} a_j z^j$ in $|z| < \rho$, let

$$P_{n-1}(z, f) := \sum_{j=0}^{n-1} a_j z^j, \qquad n = 1, 2,..., \tag{1.4}$$

be its $(n-1)$st partial sum (so that $P_{n-1}(z, f) = p_{n-1}(z, O, f)$).

With this notation, Walsh's result is

THEOREM A [4]. *For any $f \in A_\rho$, there holds*

$$\lim_{n\to\infty} \{p_{n-1}(z, E, f) - P_{n-1}(z, f)\} = 0, \qquad \text{for all } |z| < \rho^2, \tag{1.5}$$

the convergence being uniform and geometric on any closed subset of $|z| < \rho^2$. More precisely, for any r with $\rho \leqslant r < \infty$, there holds

$$\overline{\lim_{n\to\infty}} \left\{ \max_{|z|=r} |p_{n-1}(z, E, f) - P_{n-1}(z, f)| \right\}^{1/n} \leqslant r/\rho^2. \tag{1.6}$$

Further, the result of (1.5) is best possible in the sense that there is some $\hat{f} \in A_\rho$ and some $\hat{z}$ with $|\hat{z}| = \rho^2$ for which the sequence $\{p_{n-1}(\hat{z}, E, \hat{f}) - P_{n-1}(\hat{z}, \hat{f})\}_{n=1}^{\infty}$ does not tend to zero as $n \to \infty$.

Recently, Cavaretta, Sharma, and Varga [1] have generalized Walsh's Theorem A in several directions. For one of their results, define, for each positive integer l, the polynomial

$$Q_{n-1,l}(z, f) := \sum_{k=0}^{n-1} \sum_{j=0}^{l-1} a_{jn+k} z^k, \tag{1.7}$$

which is of degree at most $n-1$. Then, Walsh's Theorem A is the special case $l = 1$ of

THEOREM B [1]. *For any $f \in A_\rho$ and for any positive integer l, there holds*

$$\lim_{n\to\infty} \{p_{n-1}(z, E, f) - Q_{n-1,l}(z, f)\} = 0, \qquad \text{for all } |z| < \rho^{l+1}, \tag{1.8}$$

the convergence being uniform and geometric on any closed subset of $|z| < \rho^{l+1}$. *More precisely, for any r with* $\rho \leqslant r < \infty$, *there holds*

$$\overline{\lim_{n\to\infty}} \left\{ \max_{|z|=r} |p_{n-1}(z, E, f) - Q_{n-1,l}(z, f)| \right\}^{1/n} \leqslant r/\rho^{l+1}. \tag{1.9}$$

Further, the result of (1.8) *is best possible in the sense that there is some* $\hat{f} \in A_\rho$ *and some* $\hat{z}$ *with* $|\hat{z}| = \rho^{l+1}$ *for which the sequence* $\{p_{n-1}(\hat{z}, E, \hat{f}) - Q_{n-1,l}(\hat{z}, \hat{f})\}_{n=1}^{\infty}$ *does not tend to zero as* $n \to \infty$.

It has been conjectured by Saff and Varga that the quantity ρ^2 in (1.5) of Walsh's Theorem A is *maximal* for any interpolation matrix Z satisfying (1.2). More precisely, their conjecture is

CONJECTURE C [3, Chapter 4]. *Let* $Z = \{z_{k,n}\}$ *be any triangular interpolation matrix satisfying* (1.2). *Then, there is no* $\sigma > \rho^2$ *for which*

$$\lim_{n\to\infty} \{p_{n-1}(z, Z, f) - P_{n-1}(z, f)\} = 0,$$

$$\textit{for all } |z| < \sigma, \textit{ and for all } f \in A_\rho. \tag{1.10}$$

We remark that *some* condition on the matrix Z, such as the first inequality of (1.2), is *necessary*, as the following example shows. For any fixed $\alpha > 0$, let E_α denote the triangular interpolation matrix whose entries $z_{k,n}(\alpha)$, in its nth row, are defined to be nth roots of α^n. In this case, the analog of (1.5) of Theorem A can be verified to be

$$\lim_{n\to\infty} \{p_{n-1}(z, E_\alpha, f) - P_{n-1}(z, f)\} = 0,$$

$$\text{for all } |z| < \rho^2/\alpha, \text{ all } f \in A_\rho. \tag{1.11}$$

Obviously, $\rho^2/\alpha > \rho^2$ for any α with $0 < \alpha < 1$. Consequently, (1.10) of Conjecture C then fails for E_α when $0 < \alpha < 1$, but in this case, the interpolation points $z_{k,n}(\alpha)$ do not satisfy the first inequality of (1.2).

Actually, our first result (Theorem 1) shows that the Saff–Varga Conjecture C is *valid*, even in the more general setting of Theorem B (with ρ^{l+1} replacing ρ^2), and under a weaker hypothesis than (1.2). Specifically, consider any triangular interpolation matrix $Z = \{z_{k,n}\}$ which satisfies

$$0 \leqslant |z_{k,n}| < \rho \qquad (k = 1, 2,..., n; n = 1, 2,...). \tag{1.12}$$

Associated with the nth row of Z is the monic polynomial of degree n,

$$\omega_n(u) = \omega_n(u, Z) := \prod_{k=1}^{n} (u - z_{k,n}), \qquad n = 1, 2,.... \tag{1.13}$$

Let

$$\gamma_n(\rho, Z) := \text{modulus of the first nonzero term of} \begin{cases} \omega_n(\rho, Z), & \text{if } l > 1, \\ \omega_n(\rho, Z) - \rho^n, & \text{if } l = 1. \end{cases} \tag{1.14}$$

Note that since $\omega_n(u, Z)$ is monic, then $\gamma_n(\rho, Z)$ is well defined for all $l > 1$, and $\gamma_n(\rho, Z) > 0$ for all $n = 1, 2, \ldots$. However, if $\omega_n(u, Z) = u^n$ and if $l = 1$, then all terms of $\omega_n(\rho, Z) - \rho^n$ are zero, and $\gamma_n(\rho, Z)$ is defined to be zero in this event. Our assumption on Z, in addition to (1.12), is that

$$\mu = \mu(\rho, Z) := \overline{\lim_{n\to\infty}}\ \gamma_n^{1/n}(\rho, Z) \geqslant 1. \tag{1.15}$$

Next, as a quantitative measure for the largest disk domain of uniform and geometric convergence to zero for a particular triangular interpolation matrix Z, of the differences of the interpolating polynomials (cf. (1.3) and (1.7)) for *all* $f \in A_\rho$, we set

$$\Delta_l(r, \rho, Z) := \sup_{f\in A_\rho} \overline{\lim_{n\to\infty}} \left\{ \max_{|z|=r} |p_{n-1}(z, Z, f) - Q_{n-1,l}(z, f)| \right\}^{1/n}, \quad (r > \rho). \tag{1.16}$$

Obviously, from (1.9) of Theorem B, $\Delta_l(r, \rho, E) \leqslant r/\rho^{l+1}$, and as explicit calculations in [1, p. 158] give the reverse inequality, then

$$\Delta_l(r, \rho, E) = r/\rho^{l+1}, \qquad (r > \rho). \tag{1.17}$$

With this notation, our main result is

THEOREM 1. *Let* $Z = \{z_{k,n}\}$ *be any triangular interpolation matrix satisfying* (1.12) *and* (1.15). *Then, for each complex number* $\hat{z}$ *with*

$$|\hat{z}| > \rho^{l+1}/\mu, \tag{1.18}$$

there is an $\hat{f}$ *in* A_ρ *for which the sequence*

$$\{p_{n-1}(\hat{z}, Z, \hat{f}) - Q_{n-1,l}(\hat{z}, \hat{f})\}_{n=1}^{\infty} \tag{1.19}$$

is unbounded. In addition (cf. (1.16)), *there holds*

$$\Delta_l(r, \rho, Z) \geqslant \mu r/\rho^{l+1} \geqslant \Delta_l(r, \rho, E), \qquad \textit{for all } r > \rho. \tag{1.20}$$

The proof of Theorem 1 will be given in Section 2. Before proceeding to other results, we consider some applications of Theorem 1. First, suppose that the entries of the triangular interpolation matrix Z satisfy (1.2). As the constant term of $\omega_n(u)$ is in modulus at least unity in this case, then $\gamma_n(\rho, Z) \geqslant 1$ for all $n \geqslant 1$ and all $l \geqslant 1$, so that (1.15) is clearly satisfied. Thus, as $\mu \geqslant 1$ from (1.15), then Theorem 1 gives, for *each* complex number $\hat{z}$ with $|\hat{z}| > \rho^{l+1}$, that the sequence in (1.19) is unbounded for *some* $\hat{f}$ in A_ρ. This of course establishes the validity of Conjecture C as a special case of $l = 1$ of Theorem 1.

Continuing, it is evident that the special interpolation matrix E satisfies (1.15) with $\mu = 1$ for any $l \geqslant 1$, so that for *each* complex number $\hat{z}$ with $|\hat{z}| > \rho^{l+1}$, the sequence

$$\{p_{n-1}(\hat{z}, E, \hat{f}) - Q_{n-1,l}(\hat{z}, \hat{f})\}_{n=1}^{\infty}$$

is unbounded for *some* $\hat{f} \in A_\rho$. This should be contrasted with the recent result of Saff and Varga [2, Theorem 1] which establishes that, for *each* $f \in A_\rho$, the sequence

$$\{p_{n-1}(z, E, f) - Q_{n-1,l}(z, f)\}_{n=1}^{\infty}$$

can be bounded in *at most* l distinct points in $|z| > \rho^{l+1}$.

Next, to show that the hypothesis (1.15) can allow multiple interpolations in $|z| < 1$, suppose that the triangular interpolation matrix $\tilde{Z}$ is such that its associated polynomials (cf. (1.13)) are given by

$$\omega_n(u, \tilde{Z}) := u^{n-1}(u - \tfrac{1}{2}) = -\tfrac{1}{2}u^{n-1} + u^n, \qquad n = 1, 2, \ldots. \tag{1.21}$$

In this case,

$$\gamma_n(\rho, \tilde{Z}) = \tfrac{1}{2}\rho^{n-1}, \qquad \text{for all } n \geqslant 1, \text{ all } l \geqslant 1,$$

so that (1.15) is valid with $\mu = \rho$, for any $l \geqslant 1$. On the other hand, we see that

$$\gamma_n(\rho, E_\alpha) = \alpha^n, \qquad \text{for all } n \geqslant 1, \text{ all } l \geqslant 1,$$

so that (1.15) is *not* satisfied for any $0 < \alpha < 1$.

Next, recall that (1.20) of Theorem 1 gives that

$$\Delta_l(r, \rho, Z) \geqslant \Delta_l(r, \rho, E), \qquad \text{for all } r > \rho. \tag{1.22}$$

Our interest now is in specifying *sufficient* conditions on the matrix Z so that equality holds in (1.22) for all $r > \rho$. As we shall see in Theorem 2 below, there is a whole class of matrices Z for which equality holds in (1.22) for all $r > \rho$. Thus, for this class of matrices, one has from Theorem 1 the *optimal* disk domain of uniform and geometric convergence to zero, for the associated differences of interpolating polynomials (cf. (1.9)), for all $f \in A_\rho$.

THEOREM 2. *Let the triangular interpolation matrix* $Z = \{z_{k,n}\}$ *satisfy* (1.12) *and*

$$|z_{k,n} - \exp(2\pi ik/n)| \leqslant 1/\rho^{ln} \quad (k = 1, 2, \ldots, n;\ n = 1, 2, \ldots), \tag{1.23}$$

for some positive integer l. Then,

$$\Delta_l(r, \rho, Z) = \Delta_l(r, \rho, E) = r/\rho^{l+1}, \qquad \text{for all } r > \rho. \tag{1.24}$$

Thus, on any closed subset H of $|z| < \rho^{l+1}$, *the sequence*

$$\{p_{n-1}(z, Z, f) - Q_{n-1}(z, f)\}_{n=1}^{\infty} \tag{1.25}$$

tends to zero for all $z \in H$ *and all* $f \in A_\rho$, *while for each* $\hat{z}$ *with* $|\hat{z}| > \rho^{l+1}$, *there is an* $\hat{f} \in A_\rho$ *for which the sequence*

$$\{p_{n-1}(\hat{z}, Z, \hat{f}) - Q_{n-1,l}(\hat{z}, \hat{f})\}_{n=1}^{\infty} \tag{1.26}$$

is unbounded.

The proof of Theorem 2 will be given in Section 3. In essence, Theorem 2 states that if the interpolation points $z_{k,n}$ are sufficiently close to the nth roots of unity (cf. (1.23)), then an "optimal" interpolation matrix is obtained. For related results, see [1, Section 10].

Finally, to show that the type of assumption of (1.23) of Theorem 2 is reasonable, we include the following related result, whose proof will be given in Section 4.

THEOREM 3. *For each* $\delta > 1$, *let the triangular interpolation matrix* $Z = \{z_{k,n}\}$ satisfy (1.12) *and*

$$|z_{k,n} - \exp(2\pi ik/n)| \leqslant 1/\delta^n \qquad (k = 1, 2, \ldots, n;\ n = 1, 2, \ldots). \tag{1.27}$$

Then, for each positive integer l,

$$\Delta_l(r, \rho, Z) \leqslant \frac{r}{\rho \cdot \min(\rho^l; \delta)} \qquad (r > \rho). \tag{1.28}$$

Moreover, the inequality in (1.28) *is sharp, in that for each* $\delta > 1$, *there is a triangular interpolation matrix* $\check{Z} = \{\check{z}_{k,n}\}$ satisfying (1.12) *and* (1.27), *for which equality holds in* (1.28).

2. Proof of Theorem 1

Let

$$f_u(z) := 1/(u-z), \qquad \text{where } |u| = \rho. \tag{2.1}$$

Clearly, f_u is an element of A_ρ for any choice of the complex number u with $|u| = \rho$, and a simple computation from (1.3) and (1.7) shows that

$$p_{n-1}(z, Z, f_u) = \frac{\omega_n(u) - \omega_n(z)}{\omega_n(u)(u-z)}; \qquad Q_{n-1,l}(z, f_u) = \frac{(u^{ln}-1)(u^n - z^n)}{(u^n-1)(u-z)\,u^{ln}}, \tag{2.2}$$

for any $n \geqslant 1$. Thus, for any z with $|z| = r > \rho$, we have

$$|p_{n-1}(z, Z, f_u) - Q_{n-1,l}(z, f_u)| \geqslant \frac{|\Omega_n(u; z)|}{(r+\rho)\,\rho^{ln}\,|\omega_n(u)|}, \tag{2.3}$$

where

$$\Omega_n(u; z) := u^{ln}(\omega_n(u) - \omega_n(z)) - \left(\frac{u^{ln}-1}{u^n-1}\right)(u^n - z^n)\,\omega_n(u) \tag{2.4}$$

is a polynomial in u.

If $j(n)$ denotes the precise number of $\{z_{k,n}\}_{k=1}^n$ which are zero in the nth row of Z, then $0 \leqslant j(n) \leqslant n$, and we can write

$$\omega_n(u) := u^{j(n)}\tilde{\omega}_n(u), \qquad \text{where } \tilde{\omega}_n(0) \neq 0. \tag{2.5}$$

With (2.4) and (2.5), we can similarly write

$$\Omega_n(u; z) = u^{j(n)}\tilde{\Omega}_n(u; z), \tag{2.6}$$

where

$$\tilde{\Omega}_n(u; z) := \left\{u^{ln} - \left(\frac{u^{ln}-1}{u^n-1}\right)(u^n - z^n)\right\}\tilde{\omega}_n(u) - u^{ln-j(n)}\,z^{j(n)}\,\tilde{\omega}_n(z). \tag{2.7}$$

Obviously, from (2.5) and (2.6), we have that

$$\max_{|u|=\rho}\left|\frac{\Omega_n(u; z)}{\omega_n(u)}\right| = \max_{|u|=\rho}\left|\frac{\tilde{\Omega}_n(u; z)}{\tilde{\omega}_n(u)}\right| \qquad n = 1, 2, \ldots.$$

We next write $\tilde{\omega}_n(u) := \prod_{k=1}^{n-j(n)}(u - z'_{k,n})$, where $0 < |z'_{k,n}| < \rho$ if $j(n) < n$, and where $\tilde{\omega}_n(u) \equiv 1$ if $j(n) = n$. For any complex number $u = \rho e^{i\theta}$, θ real and arbitrary, it is evident that

$$|u - z'_{k,n}| = |\rho - z'_{k,n} e^{-i\theta}| = |\rho - \bar{z}'_{k,n} e^{i\theta}| = \left| \rho - \frac{\bar{z}'_{k,n} u}{\rho} \right|,$$

so that $|\tilde{\omega}_n(u)| = \prod_{k=1}^{n-j(n)} |\rho - (\bar{z}'_{k,n} u)/\rho|$ for all $|u| = \rho$. Setting

$$\tilde{R}_n(u; z) := \frac{\tilde{\Omega}_n(u; z)}{\prod_{k=1}^{n-j(n)} (\rho - (\bar{z}'_{k,n} u)/\rho)}, \tag{2.8}$$

then $\tilde{R}_n(u; z)$, as a function of u, is analytic in $|u| < \rho^2/(\max_k |z'_{k,n}|)$. But, as $|z'_{k,n}| < \rho$ for all $1 \leqslant k \leqslant n - j(n)$, then $\tilde{R}_n(u; z)$ is analytic in $|u| \leqslant \rho$. Thus, from the above discussion, there holds

$$\max_{|u| = \rho} \left| \frac{\Omega_n(u; z)}{\omega_n(u)} \right| = \max_{|u| = \rho} |\tilde{R}_n(u; z)|.$$

Next, with the hypothesis of (1.15), choose *any* $\hat{z}$ with

$$\mu |\hat{z}| > \rho^{l+1}, \tag{2.9}$$

and *fix* $\hat{z}$. Then, let $u_n = u_n(\hat{z})$ denote a point on $|u| = \rho$ where $|\tilde{R}_n(u; \hat{z})|$ attains its maximum. With the maximum principle and (2.8), there holds

$$\max_{|u| = \rho} \left| \frac{\Omega_n(u; \hat{z})}{\omega_n(u)} \right| = |\tilde{R}_n(u_n; \hat{z})| \geqslant |\tilde{R}_n(0; \hat{z})| = \frac{|\tilde{\Omega}_n(0; \hat{z})|}{\rho^{n-j(n)}}. \tag{2.10}$$

Now, from (2.7), it follows, with $|\hat{z}| =: r$, that

$$\begin{aligned} |\tilde{\Omega}_n(0; \hat{z})| &= r^n |\tilde{\omega}_n(0)|, && \text{when } l > 1, \\ &= r^n |\tilde{\omega}_n(0)|, && \text{when } l = 1 \text{ and } j(n) < n, \\ &= 0, && \text{when } l = 1 \text{ and } j(n) = n. \end{aligned}$$

However, from the definition in (1.14), we verify in all cases that the above can be represented as

$$|\tilde{\Omega}_n(0; \hat{z})| = \frac{r^n \gamma_n(\rho, Z)}{\rho^j(n)}. \tag{2.11}$$

Thus, combining (2.3), (2.10), and (2.11) yields

$$|p_{n-1}(\hat{z}, Z, f_{u_n}) - Q_{n-1,l}(\hat{z}, f_{u_n})| \geqslant \left(\frac{r}{\rho^{l+1}}\right)^n \left(\frac{\gamma_n(\rho, Z)}{r + \rho}\right), \tag{2.12}$$

for all $n \geqslant 1$. Again recalling the hypothesis (1.15), let ε be an arbitrary positive number such that (cf. (2.9))

$$(\mu-\varepsilon)\, r > \rho^{l+1}, \qquad \text{where } |\hat{z}| = r, \tag{2.13}$$

and let $\{n_j\}_{j=1}^{\infty}$ be an infinite sequence of positive integers with $n_1 < n_2 < \ldots$, such that

$$\gamma_{n_j}(\rho, Z) \geqslant (\mu-\varepsilon)^{n_j}, \qquad \text{for all } j \geqslant 1. \tag{2.14}$$

Thus, from (2.12) and 2.14), we obtain

$$|p_{n_j-1}(\hat{z}, Z, f_j) - Q_{n_j-1,l}(\hat{z}, f_j)| \geqslant \frac{1}{(r+\rho)} \left[\frac{(\mu-\varepsilon)\, r}{\rho^{l+1}}\right]^{n_j}, \tag{2.15}$$

for all $j \geqslant 1$, where, for convenience, we have set $f_j(z) := f_{u_{n_j}}(z)$ (cf. (2.1)). In what follows, we further define

$$\rho_n := \max_{1 \leqslant k \leqslant n} |z_{k,n}|, \qquad (\text{so that } \rho_n < \rho \text{ for all } n \geqslant 1). \tag{2.16}$$

With the positive integer n_1 of (2.14), consider $f_1(z)$ and the inequalities

$$|p_{m-1}(\hat{z}, Z, f_1) - Q_{m-1,l}(\hat{z}, f_1)| \leqslant \frac{\alpha}{\beta m} \left[\frac{(\mu-\varepsilon)\, r}{\rho^{l+1}}\right]^{m}, \qquad m > n_1, \tag{2.17}$$

where we choose any $\beta \geqslant 2$ such that

$$\beta \geqslant 1 + \frac{12(r+\rho)}{(r-\rho)} \qquad \text{and} \qquad \alpha := \frac{\beta - 1}{3(r+\rho)} > 0. \tag{2.18}$$

If the inequalities in (2.17) *fail* to hold for all m sufficiently large, there is a sequence $\{m_j\}_{j=2}^{\infty}$ of positive integers with $n_1 < m_2 < m_3 < \ldots$ for which

$$|p_{m_j-1}(\hat{z}, Z, f_1) - Q_{m_j-1,l}(\hat{z}, f_1)| > \frac{\alpha}{\beta m_j} \left[\frac{(\mu-\varepsilon)\, r}{\rho^{l+1}}\right]^{m_j}, \qquad \text{for all } j \geqslant 2.$$

This, however, would imply from (2.13) that the sequence

$$\{p_{m-1}(\hat{z}, Z, f_1) - Q_{m-1,l}(\hat{z}, f_1)\}_{m=1}^{\infty} \tag{2.19}$$

is *unbounded*, the desired result of (1.19) of Theorem 1. Otherwise, we may assume that there is an integer n_2' from the sequence $\{n_j\}_{j=1}^{\infty}$, associated with (2.14), satisfying $n_2' > n_1$, such that (2.17) holds for all $m \geqslant n_2'$, and such that

$$n_2' \geqslant \beta n_1 \left[\frac{2\rho^{l+1}}{(\mu-\varepsilon)(\rho - \rho_{n_1})}\right]^{n_1}.$$

Without loss of generality, we may assume that $n_2' = n_2$ of the sequence $\{n_j\}_{j=1}^{\infty}$. Next, considering $f_2(z)$, we similarly ask if

$$|p_{m-1}(\hat{z}, Z, f_2) - Q_{m-1,l}(\hat{z}, f_2)| \leqslant \frac{\alpha}{\beta m}\left[\frac{(\mu-\varepsilon)\, r}{\rho^{l+1}}\right]^m, \qquad m > n_2.$$

Again, if these inequalities fail to hold for all m sufficiently large, then the sequence of (2.19), with f_1 replaced by f_2, is again unbounded. Otherwise, we may assume that there is an integer n_3' from the sequence $\{n_j\}_{j=1}^{\infty}$, satisfying $n_3' > n_2$, such that the above inequality holds for all $m \geqslant n_3'$, and such that

$$n_3' \geqslant \beta n_2 \left[\frac{2\rho^{l+1}}{(\mu-\varepsilon)(\rho-\rho_{n_2})}\right]^{n_2}.$$

Again, without loss of generality, we may assume $n_3' = n_3$ in the sequence $\{n_j\}_{j=1}^{\infty}$. Continuing inductively, either the unboundedness of the sequence of (2.19) (for some f_j) is obtained after a finite number of steps, or else the infinite sequence $\{n_j\}_{j=1}^{\infty}$, associated with (2.14), satisfies

$$n_{j+1} \geqslant \beta n_j \left[\frac{2\rho^{l+1}}{(\mu-\varepsilon)(\rho-\rho_{n_j})}\right]^{n_j}, \qquad \text{for all } j \geqslant 1, \tag{2.20}$$

and for which

$$|p_{n_j-1}(\hat{z}, Z, f_k) - Q_{n_j-1,l}(\hat{z}, f_k)| \leqslant \frac{\alpha}{\beta n_j}\left[\frac{(\mu-\varepsilon)\, r}{\rho^{l+1}}\right]^{n_j}, \qquad \text{for all } j > k, \tag{2.21}$$

where $k = 1, 2, \ldots$.

Assuming (2.20) and (2.21), define

$$\hat{f}(z) := \sum_{k=1}^{\infty} \frac{f_{n_k}(z)}{n_k} = \sum_{k=1}^{\infty} \frac{1}{n_k(u_{n_k} - z)}, \tag{2.22}$$

where $|u_{n_k}| = \rho$ for all $k \geqslant 1$. Because $n_k \geqslant \beta n_{k-1}$ for any $k \geqslant 2$ from (2.20), it is evident that

$$n_k \geqslant \beta^{k-j}\, n_j, \qquad \text{for all } k \geqslant j \geqslant 1, \tag{2.23}$$

which gives that the series in (2.22) converges uniformly in $|z| < \rho$. Thus, $\hat{f}(z)$ is analytic in $|z| < \rho$. If $\hat{f}(z) := \sum_{j=0}^{\infty} \hat{a}_j z^j$, then of course the radius of convergence, R, of this Taylor expansion for $\hat{f}$ satisfies $R \geqslant \rho$. On the other hand, it follows from (2.22) that

$$\hat{a}_j = \sum_{k=1}^{\infty} \frac{1}{n_k u_{n_k}^{j+1}}, \qquad \text{for } j = 0, 1, \ldots.$$

Since $|u_{n_j}| = \rho$, then

$$|\hat{a}_j| \geqslant \frac{1}{\rho^{j+1}} \left\{ \frac{1}{n_1} - \sum_{k=2}^{\infty} \frac{1}{n_k} \right\}.$$

With the inequalities of (2.23), it easily follows that

$$|\hat{a}_j| \geqslant \frac{(\beta - 2)}{(\beta - 1)\, n_1 \rho^{j+1}}, \qquad \text{for } j = 0, 1, \ldots,$$

so that

$$\overline{\lim_{n \to \infty}}\, |\hat{a}_j|^{1/j} \geqslant 1/\rho.$$

This implies that $R \leqslant \rho$, and as the reverse inequality was established above, then $R = \rho$. Consequently, $\hat{f}$ has a singularity on the circle $|z| = \rho$, and $\hat{f}$ is an element of A_ρ.

From the linearity of the operators involved and the triangle inequality, we have, from (2.22), that

$$|p_{n_j-1}(\hat{z}, Z, \hat{f}) - Q_{n_j-1,l}(\hat{z}, \hat{f})| \geqslant S_1 - S_2 - S_3 - S_4, \tag{2.24}$$

where

$$S_1 := \frac{1}{n_j} |p_{n_j-1}(\hat{z}, Z, f_j) - Q_{n_j-1,l}(\hat{z}, f_j)|,$$

$$S_2 := \sum_{k=1}^{j-1} \frac{1}{n_k} |p_{n_j-1}(\hat{z}, Z, f_k) - Q_{n_j-1,l}(\hat{z}, f_k)|,$$

$$S_3 := \sum_{k=j+1}^{\infty} \frac{1}{n_k} |p_{n_j-1}(\hat{z}, Z, f_k)|,$$

and

$$S_4 := \sum_{k=j+1}^{\infty} \frac{1}{n_k} |Q_{n_j-1,l}(\hat{z}, f_k)|.$$

From (2.15), we have

$$S_1 \geqslant \frac{1}{(r+\rho)\, n_j} \left[\frac{(\mu - \varepsilon)\, r}{\rho^{l+1}} \right]^{n_j}, \qquad \text{for all } j \geqslant 1, \tag{2.25}$$

while from (2.21) we have

$$S_2 \leqslant \frac{\alpha}{\beta n_j} \left[\frac{(\mu - \varepsilon)\, r}{\rho^{l+1}} \right]^{n_j} \sum_{k=1}^{j-1} \frac{1}{n_k}.$$

Applying the inequalities of (2.23) to the above sum yields

$$S_2 \leqslant \frac{\alpha}{(\beta-1)\,n_j}\left[\frac{(\mu-\varepsilon)\,r}{\rho^{l+1}}\right]^{n_j}, \qquad \text{for all } j \geqslant 2. \tag{2.26}$$

Next, from the first equation of (2.2), from the definition of ρ_n in (2.16), and from the fact that $|u_{n_j}| = \rho$, it readily follows that

$$|p_{n_j-1}(\hat{z}, Z, f_{n_k})| \leqslant \frac{(\rho+\rho_{n_j})^{n_j} + (r+\rho_{n_j})^{n_j}}{(\rho-\rho_{n_j})^{n_j}(r-\rho)}, \qquad \text{for all } k \geqslant j+1.$$

Since $\rho + \rho_{n_j} < 2r$, and as $r + \rho_{n_j} < 2r$, the above implies that

$$|p_{n_j-1}(\hat{z}, Z, f_{n_k})| \leqslant \frac{2(2r)^{n_j}}{(\rho-\rho_{n_j})^{n_j}(r-\rho)}, \qquad \text{for all } k \geqslant j+1. \tag{2.27}$$

Similarly, from the second equation of (2.2), we deduce

$$|Q_{n_j-1,l}(\hat{z}, Z, f_{n_k})| \leqslant \frac{(1+\rho^{-ln_j})(\rho^{n_j}+r^{n_j})}{(\rho^{n_j}-1)(r-\rho)} \leqslant \frac{4r^{n_j}}{(\rho^{n_j}-1)(r-\rho)}.$$

Since $\rho > \rho_n \geqslant 0$ and since $\rho > 1$, then $(\rho-\rho_n)^n \leqslant \rho^n \leqslant 2(\rho^n - 1)$ for all n sufficiently large, so that

$$|Q_{n_j-1,l}(\hat{z}, Z, f_{n_k})| \leqslant \frac{8r^{n_j}}{(\rho-\rho_{n_j})^{n_j}(r-\rho)}, \qquad \text{for all } k \geqslant j+1, \text{ all } j \text{ large.} \tag{2.28}$$

Thus,

$$S_3 + S_4 \leqslant \frac{4(2r)^{n_j}}{(\rho-\rho_{n_j})^{n_j}(r-\rho)} \sum_{k=j+1}^{\infty} \frac{1}{n_k}, \qquad \text{for all } k \geqslant 1, \text{ all } j \text{ large.}$$

Again using (2.23) and (2.20),

$$\sum_{k=j+1}^{\infty} \frac{1}{n_k} \leqslant \frac{\beta}{(\beta-1)\,n_{j+1}} \leqslant \frac{1}{(\beta-1)\,n_j}\left[\frac{(\mu-\varepsilon)(\rho-\rho_{n_j})}{2\rho^{l+1}}\right]^{n_j}$$

so that

$$S_3 + S_4 \leqslant \frac{4}{(r-\rho)(\beta-1)\,n_j}\left[\frac{(\mu-\varepsilon)\,r}{\rho^{l+1}}\right]^{n_j}, \qquad \text{for all } j \text{ sufficiently large.}$$

On combining these inequalities and using the definitions of (2.18), we see that

$$S_1 - S_2 - S_3 - S_4 \geqslant \frac{1}{3(r+\rho)\,n_j}\left[\frac{(\mu-\varepsilon)\,r}{\rho^{l+1}}\right]^{n_j}, \qquad \text{for all } j \text{ sufficiently large,}$$

which implies from (2.24) that

$$|p_{n_j-1}(\hat{z}, Z, \hat{f}) - Q_{n_j-1,l}(\hat{z}, \hat{f})| \geqslant \frac{1}{3(r+\rho)\, n_j} \left[\frac{(\mu-\varepsilon)\, r}{\rho^{l+1}}\right]^j, \tag{2.29}$$

for all j sufficiently large. Thus, from (2.13), we deduce that the sequence

$$\{p_{n-1}(\hat{z}, Z, \hat{f}) - Q_{n-1,l}(\hat{z}, \hat{f})\}_{n=1}^{\infty}$$

is unbounded, the desired result (1.19) of Theorem 1.

To conclude the proof of Theorem 1, we simply note that the above construction is valid for *any* choice of the complex number $\hat{z}$ with $|\hat{z}| = r > \rho$, and any ε with $0 < \varepsilon < \mu$, so that (2.29) holds, in particular, for any $|\hat{z}| = r > \rho$. Thus,

$$\overline{\lim_{n\to\infty}} \left\{\max_{|z|=r} |p_{n-1}(z, Z, \hat{f}) - Q_{n-1,l}(z, \hat{f})|\right\}^{1/n} \geqslant \frac{(\mu-\varepsilon)\, r}{\rho^{l+1}}, \qquad (r > \rho),$$

and as $\hat{f}$ is an element of A_ρ, then from (1.16)

$$\Delta_l(r, \rho, Z) \geqslant \frac{(\mu-\varepsilon)\, r}{\rho^{l+1}}, \qquad \text{for any } r > \rho.$$

As $\varepsilon > 0$ is arbitrary, we thus have, with (1.17),

$$\Delta_l(r, \rho, Z) \geqslant \frac{\mu r}{\rho^{l+1}} \geqslant \Delta_l(r, \rho, E), \qquad \text{for all } r > \rho, \tag{2.30}$$

which is the desired result of (1.20) of Theorem 1. ∎

3. Proof of Theorem 2

With the assumption of (1.23), we have that

$$1 + \frac{1}{\rho^{ln}} \geqslant |z_{k,n}| \geqslant 1 - \frac{1}{\rho^{ln}}, \qquad \text{for all } k = 1, 2, \ldots, n;\ n = 1, 2, \ldots,$$

so that

$$\left(1 + \frac{1}{\rho^{ln}}\right)^n \geqslant \prod_{k=1}^{n} |z_{k,n}| \geqslant \left(1 - \frac{1}{\rho^{ln}}\right)^n, \qquad \text{for all } n = 1, 2, \ldots. \tag{3.1}$$

By definition (1.14), it follows that $\gamma_n(\rho, Z) = \prod_{k=1}^{n} |z_{k,n}|$, so that from (3.1),

$$\lim_{n\to\infty} \gamma_n^{1/n}(\rho, Z) = 1. \tag{3.2}$$

Consequently, as (1.15) is thus valid with $\mu = 1$, we have from (1.20) of Theorem 1 that

$$\Delta_l(r, \rho, Z) \geqslant \frac{r}{\rho^{l+1}} = \Delta_l(r, \rho, E), \qquad \text{for all } r > \rho. \tag{3.3}$$

We next establish the reverse inequality of (3.3).

From Hermite's integral representation, there holds

$$p_{n-1}(z, Z, f) = \frac{1}{2\pi i} \int_\Gamma \left(\frac{f(t)}{t-z}\right)\left[\frac{\omega_n(t, Z) - \omega_n(z, Z)}{\omega_n(t, Z)}\right] dt,$$

$$\text{for any } f \in A_\rho;$$

here, $\Gamma := \{t : |t| = R\}$, where $\rho_n < R < \rho$. Note that since $\rho_n \leqslant 1 + \rho^{-ln}$ from hypothesis (1.23), then R can be chosen arbitrarily in $1 < R < \rho$, for all n sufficiently large. Next, since $\omega_n(t, E) = t^n - 1$ for $n = 1, 2, \ldots$, it similarly follows (cf. [1, p. 157]) that

$$p_{n-1}(z, E, f) - Q_{n-1,l}(z, f) = \frac{1}{2\pi i} \int_\Gamma \left(\frac{f(t)}{t-z}\right)\left[\frac{t^n - z^n}{(t^n - 1)\, t^{ln}}\right] dt =: I_1, \tag{3.5}$$

and that

$$p_{n-1}(z, Z, f) - Q_{n-1,l}(z, f) = I_1 + I_2, \tag{3.6}$$

where

$$I_2 := \frac{1}{2\pi i} \int_\Gamma \left(\frac{f(t)}{t-z}\right)\left(\frac{z^n - 1}{t^n - 1}\right)\left[1 - \left(\frac{t^n - 1}{\omega_n(t, Z)}\right)\left(\frac{\omega_n(z, Z)}{z^n - 1}\right)\right] dt. \tag{3.7}$$

Now, by definition, we can write that

$$\begin{aligned} T &:= \left|\left(\frac{t^n - 1}{\omega_n(t, Z)}\right)\left(\frac{\omega_n(z, Z)}{z^n - 1}\right) - 1\right| \\ &= \left|\prod_{k=1}^{n}\left(\frac{t - \exp(2\pi i k/n)}{t - z_{k,n}}\right)\left(\frac{z - z_{k,n}}{z - \exp(2\pi i k/n)}\right) - 1\right| \\ &= \left|\prod_{k=1}^{n}\left\{1 + \frac{z_{k,n} - \exp(2\pi i k/n)}{t - z_{k,n}}\right\} \cdot \left\{1 + \frac{\exp(2\pi i k/n) - z_{k,n}}{z - \exp(2\pi i k/n)}\right\} - 1\right|. \end{aligned}$$

With hypothesis (1.23), we have $|z_{k,n} - \exp(2\pi ik/n)| \leqslant \rho^{-ln}$, while $|t - z_{k,n}| \geqslant R - \rho_n$ and $|z - \exp(2\pi ik/n)| \geqslant r - 1 > R - \rho_n$, for all n sufficiently large, where $|z| = r > \rho$. Hence,

$$T \leqslant \left(1 + \frac{1}{\rho^{ln}(R - \rho_n)}\right)^{2n} - 1 \leqslant \frac{6n}{\rho^{ln}(R - \rho_n)}, \tag{3.8}$$

for all n sufficiently large. Thus, if $M := \max_{|t|=R} |f(t)|$, the integral I_2 in (3.7) is bounded above in modulus, using (3.8), by

$$|I_2| \leqslant \frac{M \cdot R}{(r - R)} \left(\frac{r^n + 1}{R^n - 1}\right) \frac{6n}{\rho^{ln}(R - \rho_n)}, \qquad \text{for all } n \text{ sufficiently large,} \tag{3.9}$$

while the integral I_1 in (3.5) is similarly bounded above in modulus by

$$|I_1| \leqslant \frac{M \cdot R}{(r - R)} \left(\frac{r^n + R^n}{(R^n - 1)\, R^{ln}}\right). \tag{3.10}$$

Hence, from (3.6), (3.9), and (3.10), it easily follows that

$$\overline{\lim_{n=\infty}} \left\{\max_{|z|=r} |p_{n-1}(z, Z, f) - Q_{n-1,l}(z, f)|\right\}^{1/n} \leqslant \frac{r}{R^{(l+1)n}},$$

but as the left side is independent of the choice of R in $1 < R < \rho$, we can let R approach ρ, giving

$$\overline{\lim_{n\to\infty}} \left\{\max_{|z|=r} |p_{n-1}(z, Z, f) - Q_{n-1,l}(z, f)|\right\}^{1/n} \leqslant \frac{r}{\rho^{(l+1)n}},$$

for any $r > \rho$ and any $f \in A_\rho$. Consequently, from the definition in (1.16),

$$\Delta_l(r, \rho, Z) \leqslant \frac{r}{\rho^{(l+1)n}}, \qquad \text{for all } r > \rho. \tag{3.11}$$

Thus, with (3.3), we have

$$\Delta_l(r, \rho, Z) = \frac{r}{\rho^{(l+1)n}}, \qquad \text{for all } r > \rho, \tag{3.12}$$

the desired result of (1.24) of Theorem 2, and (3.12) directly gives (1.25) of Theorem 2. Finally, as $\mu = 1$ from (3.2), then (1.26) follows directly from (1.19) of Theorem 1. ∎

4. Proof of Theorem 3

If the triangular interpolation matrix $Z = \{z_{k,n}\}$ satisfies (1.12) and (1.27) with $\delta \geqslant \rho^l$, then (1.24) of Theorem 2 gives that

$$\Delta_l(r, \rho, Z) = \frac{r}{\rho^{l+1}} = \frac{r}{\rho \cdot \min(\rho^l; \delta)} \qquad (r > \rho),$$

which gives a stronger form of the desired result of Theorem 3. Thus, we may assume in what follows that δ satisfies $1 < \delta < \rho^l$.

On similarly using the integral representation of (3.4) and the definitions in (3.5)–(3.7) from the proof of Theorem 2, it easily follows that the hypothesis of (1.27) of Theorem 3 yields that $\rho_n \leqslant 1 + \delta^{-n}$ and that (cf. (3.8))

$$T \leqslant \left(1 + \frac{1}{\delta^n (R - \rho_n)}\right)^{2n} - 1 \leqslant \frac{6n}{\delta^n (R - \rho_n)} \tag{4.1}$$

for all n sufficiently large, where R can be chosen arbitrarily in $1 < R < \rho$. Similarly (cf. (3.9)),

$$|I_2| \leqslant \frac{M \cdot R}{(r - R)} \left(\frac{r^n + 1}{R^n - 1}\right) \frac{6n}{\delta^n (R - \rho_n)} \tag{4.2}$$

for all n sufficiently large, and (cf. (3.10))

$$|I_1| \leqslant \frac{M \cdot R}{(r - R)} \left(\frac{r^n + R^n}{(R^n - 1) R^{ln}}\right). \tag{4.3}$$

Thus, as in the proof of Theorem 2, it easily follows that since $1 < \delta < \rho^l$,

$$\varlimsup_{n \to \infty} \left\{\max_{|z| = r} |p_{n-1}(z, Z, f) - Q_{n-1,l}(z, f)|\right\}^{1/n} \leqslant \frac{r}{\rho\delta}$$

for any $r > \rho$ and for any $f \in A_\rho$. Consequently, from the definition in (1.16),

$$\Delta_l(r, \rho, Z) \leqslant \frac{r}{\rho\delta}, \tag{4.4}$$

the desired result of (1.28) of Theorem 3 when $1 < \delta < \rho^l$.

Finally, to show that equality can hold in (4.4), define the triangular interpolation matrix $\check{Z} = \{\check{z}_{k,n}\}$ by means of (cf. (1.13))

$$\omega_n(z, \check{Z}) := \left(\frac{z - e^{i\delta^{-n}}}{z - 1}\right)(z^n - 1), \qquad n = 1, 2, \ldots, \tag{4.5}$$

so that $\check{Z}$ clearly satisfies (1.12) and (1.27). With $\check{f}(z) := (\rho - z)^{-1}$, an element of A_ρ, we have

$$|p_{n-1}(r, \check{Z}, \check{f}) - Q_{n-1,l}(r, \check{f})| \geqslant |p_{n-1}(r, \check{Z}, \check{f}) - p_{n-1}(r, E, \check{f})| \\ - |p_{n-1}(r, E, \check{f}) - Q_{n-1,l}(r, \check{f})| =: V_1 - V_2. \qquad (4.6)$$

Next, as the interpolation polynomial $p_{n+1}(z, Z, \check{f})$ of (1.3) can be expressed as

$$p_{n-1}(z, Z, \check{f}) = \frac{\omega_n(\rho, Z) - \omega_n(z, Z)}{(\rho - z)\, \omega_n(\rho, Z)} = \frac{1}{\rho - z} - \frac{\omega_n(z, Z)}{(\rho - z)\, \omega_n(\rho, Z)}$$

for *any* triangular interpolation matrix Z satisfying (1.12), then

$$V_1 := |p_{n-1}(r, \check{Z}, \check{f}) - p_{n-1}(r, E, \check{f})| = \frac{1}{(r-\rho)} \left| \frac{\omega_n(r, \check{Z})}{\omega_n(\rho, \check{Z})} - \frac{r^n - 1}{\rho^n - 1} \right|$$

for any $r > \rho$, so that with (4, 5),

$$V_1 := \frac{(r^n - 1)}{(r-\rho)(\rho^n - 1)} \left| \frac{(\rho - 1)(r - e^{i\delta^{-n}})}{(r-1)(\rho - e^{i\delta^{-n}})} - 1 \right| = \frac{(r^n - 1)|e^{i\delta^{-n}} - 1|}{(\rho^n - 1)(r-1)|\rho - e^{i\delta^{-n}}|}$$
$$\geqslant \frac{(r^n - 1)}{\delta^n(\rho^n - 1)(r-1)(\rho+1)} = \left(\frac{r}{\rho\delta}\right)^n \left\{ \frac{1 - r^{-n}}{(1-\rho^{-n})(r-1)(\rho+1)} \right\},$$

whence

$$V_1 \geqslant \left(\frac{r}{\rho\delta}\right)^n \cdot \frac{(1/2)}{(r-1)(\rho+1)}, \qquad \text{for all } n \geqslant n_1(r, \rho). \qquad (4.7)$$

Similarly, using (2.6) of [1], it follows that

$$p_{n-1}(z, E, \check{f}) - Q_{n-1,l}(z, \check{f}) = \frac{\rho^n - z^n}{(\rho - z)(\rho^n - 1)\, \rho^{ln}},$$

so that

$$V_2 := |p_{n-1}(r, E, \check{f}) - Q_{n-1,l}(r, \check{f})| = \frac{r^n - \rho^n}{(r-\rho)(\rho^n - 1)\, \rho^{ln}},$$

whence

$$V_2 \leqslant \left(\frac{r}{\rho^{l+1}}\right)^n \cdot \frac{2}{(r-\rho)}, \qquad \text{for all } n \geqslant n_2(r, \rho). \qquad (4.8)$$

Using (4.7) and (4.8) and recalling that $1 < \delta < \rho^l$, it follows from (4.6) that

$$|p_{n-1}(r, \check{Z}, \check{f}) - Q_{n-1}(r, \check{f})| \geqslant \left(\frac{r}{\rho\delta}\right)^n \cdot \frac{1/4}{(r-1)(\rho+1)}, \quad \text{for all } n \geqslant n_3(r, \rho), \tag{4.9}$$

which implies (cf. (1.16)) that

$$\Delta_l(r, \rho, \check{Z}) \geqslant \frac{r}{\rho\delta} \qquad (r > \rho). \tag{4.10}$$

As the reverse inequality holds from (4.4), then

$$\Delta_l(r, \rho, \check{Z}) = \frac{r}{\rho\delta}, \tag{4.11}$$

which establishes the desired sharpness in (1.28) of Theorem 3. ∎

REFERENCES

1. A. S. CAVARETTA, JR., A. SHARMA, AND R. S. VARGA, Interpolation in the roots of unity: An extension of a Theorem of J. L. Walsh, *Resultate Math.* **3** (1981), 155–191.
2. E. B. SAFF AND R. S. VARGA, A note on the sharpness of J. L. Walsh's Theorem and its extensions for interpolation in the roots of unity, *Studia Sci. Math. Hungar.*, in press.
3. R. S. VARGA, Topics in polynomial and rational interpolation and approximation, in "Séminaire de Mathématiques Supérieures," Vol. 81, Les Presses de l'Université de Montréal, 1982.
4. J. L. WALSH,, "Interpolation and Approximation by Rational Functions in the Complex Domain," 5th ed., Vol. XX, American Mathematical Society Colloquium Publications, Providence, R.I., 1969.

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