SEPARATUM

OF J. L. WALSH'S THEOREM AND ITS EXTENSION FOR INTERPOLATION IN THE ROOTS OF UNITY

E. B. SAFF[1] (Tampa) and R. S. VARGA[2] (Kent)

§ 1. Introduction and statements of new results

Let A_ϱ denote the collection of functions analytic in $|z|<\varrho$ and having a sir larity on the circle $|z|=\varrho$, where it is assumed that $1<\varrho<\infty$. Next, for each posi integer n, let $p_{n-1}(z; f)$ denote the Lagrange polynomial interpolant, of degre most $n-1$, of $f(z)\in A_\varrho$ in the n-th roots of unity, i.e.,

(1.1) $$p_{n-1}(\omega; f) = f(\omega)$$

where ω is any n-th root of unity, and let

(1.2) $$P_{n-1}(z; f) := \sum_{k=0}^{n-1} a_k z^k$$

AKADÉMIAI KIADÓ, BUDAPEST
PUBLISHING HOUSE OF THE HUNGARIAN ACADEMY OF SCIENCES
VERLAG DER UNGARISCHEN AKADEMIE DER WISSENSCHAFTEN
MAISON D'EDITIONS DE L'ACADEMIE DES SCIENCES DE HONGRIE
ИЗДАТЕЛЬСТВО АКАДЕМИИ НАУК ВЕНГРИИ

Acta Math. Hung.
41 (3—4) (1983), 371—377.

A NOTE ON THE SHARPNESS OF J. L. WALSH'S THEOREM AND ITS EXTENSIONS FOR INTERPOLATION IN THE ROOTS OF UNITY

E. B. SAFF[1] (Tampa) and R. S. VARGA[2] (Kent)

§ 1. Introduction and statements of new results

Let A_ϱ denote the collection of functions analytic in $|z|<\varrho$ and having a singularity on the circle $|z|=\varrho$, where it is assumed that $1<\varrho<\infty$. Next, for each positive integer n, let $p_{n-1}(z; f)$ denote the Lagrange polynomial interpolant, of degree at most $n-1$, of $f(z)\in A_\varrho$ in the n-th roots of unity, i.e.,

$$p_{n-1}(\omega; f) = f(\omega) \tag{1.1}$$

where ω is any n-th root of unity, and let

$$P_{n-1}(z; f) := \sum_{k=0}^{n-1} a_k z^k \tag{1.2}$$

be the $(n-1)$-st partial sum of $f(z)=\sum_{k=0}^{\infty} a_k z^k$. Letting

$$D_\tau := \{z\in \mathbf{C}:\ |z| < \tau\}, \tag{1.3}$$

then a beautiful result of J. L. Walsh [2, p. 153] can be stated as

THEOREM A. *For each $f(z)\in A_\varrho$, the interpolating polynomials of* (1.1) *and* (1.2) *satisfy*

$$\lim_{n\to\infty} \{p_{n-1}(z; f)-P_{n-1}(z; f)\} = 0, \quad \textit{for all} \quad z\in D_{\varrho^2}. \tag{1.4}$$

Moreover, the result of (1.4) *is best possible in the sense that there is some $\hat f(z)\in A_\varrho$ and some $\hat z$ with $|\hat z|=\varrho^2$ for which the sequence $\{p_{n-1}(\hat z; \hat f)-P_{n-1}(\hat z; \hat f)\}_{n=1}^{\infty}$ does not tend to zero as $n\to\infty$.*

Note that in Theorem A, *no* sharpness assertions are made for *arbitrary* functions $f(z)\in A_\varrho$; in particular, no statement is made on the behavior of the sequence

$$\{p_{n-1}(z; f)-P_{n-1}(z; f)\}_{n=1}^{\infty} \tag{1.5}$$

in $|z|>\varrho^2$. One of the aims of this note is to in fact address this behavior in $|z|>\varrho^2$. As a special case of Theorem 1 below, we prove that, for *any* $f(z)\in A_\varrho$, the sequence in (1.5) can be bounded in at most *one* point in $|z|>\varrho^2$. This fact is of special interest in the case when $f(z)$ in A_ϱ is also continuous in the disk $|z|\leqq\varrho$; for such functions, it has been shown in [1, Thm. 2] that (1.4) is valid for *all* $|z|\leqq\varrho^2$.

[1] Research supported in part by the National Science Foundation.
[2] Research supported in part by the Air Force Office of Scientific Research, and by the Department of Energy.

For our own purposes below, we need a recent extension of Theorem A. For additional notation, set

$$P_{n-1,j}(z;f) := \sum_{k=0}^{n-1} a_{k+jn} z^k, \quad j = 0, 1, \ldots. \tag{1.6}$$

Then, the following result of Cavaretta, Sharma, and Varga [1, Thm. 1], which gives Theorem A as the special case $l=1$, can be stated as

THEOREM B. *For each $f(z)\in A_\varrho$, and for each positive integer l, there holds*

$$\lim_{n\to\infty}\left\{p_{n-1}(z;f)-\sum_{j=0}^{l-1} P_{n-1,j}(z;f)\right\}=0, \quad \textit{for all} \quad z\in D_{\varrho^{l+1}}, \tag{1.7}$$

the convergence being uniform and geometric on any closed subset of $D_{\varrho^{l+1}}$. Moreover, the result of (1.7) *is best possible in the sense that there is some $\tilde{f}(z)\in A_\varrho$ and some $\tilde{z}$ with $|\tilde{z}|=\varrho^{l+1}$ for which the sequence*

$$\left\{p_{n-1}(z;f)-\sum_{j=0}^{l-1} P_{n-1,j}(z;f)\right\}_{n=1}^{\infty} \tag{1.8}$$

with $z=\tilde{z}$ and $f=\tilde{f}$, does not tend to zero as $n\to\infty$.

Our first new result is

THEOREM 1. *For each $f(z)\in A_\varrho$, and for each positive integer l, the sequence* (1.8) *can be bounded in at most l distinct points in $|z|>\varrho^{l+1}$. This result is sharp, in the sense that, given any l distinct points $\{\eta_k\}_{k=1}^{l}$ in the annulus $\varrho^{l+1}<|z|<\varrho^{l+2}$, there is an $\hat{f}(z)\in A_\varrho$ for which*

$$\lim_{n\to\infty}\left\{p_{n-1}(\eta_k;\hat{f})-\sum_{j=0}^{l-1} P_{n-1,j}(\eta_k;\hat{f})\right\}=0, \quad k=1,2,\ldots,l. \tag{1.9}$$

There is an extension of Theorem 1 which we can also state. Note, of course, that Theorem A involves only the Lagrange interpolation of f in the n-th roots of unity. For r a fixed positive integer, Theorem B can be extended using Hermite interpolation. For notation, let $h_{rn-1}(z;f)$ denote the Hermite polynomial interpolant, of degree at most $rn-1$, to $f, f', \ldots, f^{(r-1)}$ in the n-th roots of unity, i.e.,

$$h_{rn-1}^{(j)}(\omega;f)=f^{(j)}(\omega), \quad j=0,1,\ldots,r-1, \tag{1.10}$$

where again ω is any n-th root of unity. If $f(z)=\sum_{j=0}^{\infty} a_j z^j$, we set

$$H_{rn-1,0}(z;f) := \sum_{k=0}^{rn-1} a_k z^k, \tag{1.11}$$

and we set

$$H_{rn-1,j}(z;f) := \hat{\beta}_j(z^n)\sum_{k=0}^{n-1} a_{k+n(r+j-1)} z^k, \quad j=1,2,\ldots, \tag{1.12}$$

where

$$\hat{\beta}_j(z) := \sum_{k=0}^{r-1} \binom{r+j-1}{k} (z-1)^k, \quad j=1,2,\ldots. \tag{1.13}$$

Then, the following result of Cavaretta, Sharma, and Varga [1, Thm. 3], which gives Theorem B as the special case $r=1$, can be stated as

THEOREM C. *For each* $f(z)\in A_\varrho$, *and for each pair of positive integers* r *and* l, *there holds*

$$\lim_{n\to\infty}\left\{h_{rn-1}(z;\, f)-\sum_{j=0}^{l-1} H_{rn-1,j}(z;\, f)\right\}=0, \quad \text{for all} \quad z\in D_{\varrho^{1+(l/r)}}, \tag{1.14}$$

the convergence being uniform and geometric for any closed subset of $D_{\varrho^{1+(l/r)}}$. *Moreover, the result of* (1.14) *is best possible in the sense that there is some* $\hat{f}(z)\in A_\varrho$ *and some* $\hat{z}$ *with* $|\hat{z}|=\varrho^{1+(l/r)}$ *for which the sequence*

$$\left\{h_{rn-1}(z;\, f)-\sum_{j=0}^{l-1} H_{rn-1,j}(z;\, f)\right\}_{n=1}^{\infty}, \tag{1.15}$$

with $z=\hat{z}$ *and* $f=\hat{f}$, *does not tend to zero as* $n\to\infty$.

Our second new result, which sharpens Theorem C and gives Theorem 1 as the special case $r=1$, can be stated as

THEOREM 2. *For each* $f(z)\in A_\varrho$, *and for each pair of positive integers* r *and* l, *the sequence* (1.15) *can be bounded in at most* $r+l-1$ *distinct points in* $|z|>\varrho^{1+(l/r)}$. *This result is sharp, in the sense that, given any* $r+l-1$ *distinct points* $\{\eta_k\}_{k=1}^{r+l-1}$ *in the annulus* $\varrho^{1+(l/r)}<|z|<\min\left\{\varrho^{l+2};\, \varrho^{1+\frac{l}{r-1}}\right\}$, *there is an* $\tilde{f}(z)\in A_\varrho$ *for which*

$$\lim_{n\to\infty}\left\{h_{rn-1}(\eta_k;\, \tilde{f})-\sum_{j=0}^{l-1} H_{rn-1,j}(\eta_k;\, \tilde{f})\right\}=0, \quad k=1,2,\dots,r+l-1. \tag{1.16}$$

Since the proof of Theorem 2 is completely analogous to the proof of Theorem 1, we shall give only the proof of Theorem 1.

§ 2. Proof of Theorem 1

To establish the first part of Theorem 1, consider any (fixed $f\in A_\varrho$, consider any fixed positive integer l, and suppose that there are $(l+1)$ distinct points $\{y_k\}_{k=1}^{l+1}$ in $|z|>\varrho^{l+1}$ for which

$$\left|p_{n-1}(y_k;\, f)-\sum_{j=0}^{l-1} P_{n-1,j}(y_k;\, f)\right|\leqq M, \quad \forall n\geqq 1, \quad \forall 1\leqq k\leqq l+1. \tag{2.1}$$

If $f(z):=\sum_{j=0}^{\infty} a_j z^j$, then the hypothesis that f is analytic in $|z|<\varrho$ with a singularity on $|z|=\varrho$ gives us that

$$\overline{\lim_{n\to\infty}}\, |a_n|^{1/n}=\frac{1}{\varrho}. \tag{2.2}$$

Thus, for any $\varepsilon>0$ with $1<\varrho-\varepsilon$ and with

$$(\varrho-\varepsilon)^{l+2} > \varrho^{l+1}, \tag{2.3}$$

there is an $n_0(\varepsilon)$ for which

$$|a_n| \leqq \frac{1}{(\varrho-\varepsilon)^n}, \quad \forall n \geqq n_0(\varepsilon). \tag{2.4}$$

Next, since all the points $\{y_k\}_{k=1}^{l+1}$ lie in $|z|>\varrho^{l+1}$, then

$$\varrho^{l+1} < \sigma_1 := \min_{1\leqq k\leqq l+1} |y_k| \leqq \max_{1\leqq k\leqq l+1} |y_k| =: \sigma_2, \tag{2.5}$$

and we choose the least positive integer m for which

$$\sigma_2 < \varrho^{m+1}, \quad (\text{where } l < m). \tag{2.6}$$

Applying Theorem B (with l chosen as m), we have that the sequence $\left\{p_{n-1}(z;f)-\sum_{j=0}^{m-1} P_{n-1,j}(z;f)\right\}_{n=1}^{\infty}$ converges to zero for all $z\in D_{\varrho^{m+1}}$. In particular, as the points $\{y_k\}_1^{l+1}$ all lie in $D_{\varrho^{m+1}}$ from (2.5) and (2.6), then there exists a constant M_1 such that

$$\left|p_{n-1}(y_k;f)-\sum_{j=0}^{m-1} P_{n-1,j}(y_k;f)\right| \leqq M_1, \quad \forall n\geqq 1, \ \forall 1\leqq k\leqq l+1. \tag{2.7}$$

Using the hypothesis of (2.1), this in turn implies that

$$\left|\sum_{j=l}^{m-1} P_{n-1,j}(y_k;f)\right| \leqq M_2, \quad \forall n\geqq 1, \ \forall 1\leqq k\leqq l+1. \tag{2.8}$$

Recalling from (1.6) the definition of $P_{n-1,j}(z;f)$, then it follows from (2.4) that

$$|P_{n-1,j}(z;f)| \leqq \sum_{k=0}^{n-1} \frac{|z|^k}{(\varrho-\varepsilon)^{k+jn}} = \frac{1}{(\varrho-\varepsilon)^{jn}} \sum_{k=0}^{n-1} \left(\frac{|z|}{\varrho-\varepsilon}\right)^k, \quad \forall n \geqq n_0(\varepsilon).$$

Thus,

$$|P_{n-1,j}(z;f)| \leqq \frac{n|z|^n}{(\varrho-\varepsilon)^{(j+1)n}}, \quad \forall n\geqq n_0(\varepsilon), \ \forall |z|>\varrho, \ \forall j\geqq 1. \tag{2.9}$$

This can be used as follows. From (2.9), we see that, if $l+1\leqq m-1$, then

$$\left|\sum_{j=l+1}^{m-1} P_{n-1,j}(z;f)\right| \leqq \frac{(m-l-1)n|z|^n}{(\varrho-\varepsilon)^{(l+2)n}}, \quad \forall n\geqq n_0(\varepsilon), \ \forall |z|>\varrho. \tag{2.10}$$

Hence, from (2.8) and (2.10),

$$|P_{n-1,l}(y_k;f)| \leqq M_2+\frac{(m-l-1)n|y_k|^n}{(\varrho-\varepsilon)^{(l+2)n}}, \quad \forall n\geqq n_0(\varepsilon), \ \forall 1\leqq k\leqq l+1. \tag{2.11}$$

Now, because of (2.11), it further follows that

$$|y_k^l P_{n,l}(y_k;\, f) - P_{n-1,l}(y_k;\, f)| \leqq M_3 + \frac{M_4 n |y_k|^n}{(\varrho-\varepsilon)^{(l+2)n}}, \tag{2.12}$$

for all $n \geqq n_0(\varepsilon)$, all $1 \leqq k \leqq l+1$. Next, because of the definition of $P_{n-1,j}(z;\, f)$, it can be verified that

$$z^l P_{n,l}(z;\, f) - P_{n-1,l}(z;\, f) = \sum_{j=n}^{l+n} a_{ln+j} z^j - \sum_{j=0}^{l-1} a_{ln+j} z^j. \tag{2.13}$$

Obviously, the last term in (2.13) is bounded, independent of n, in the points $\{y_k\}_{k=1}^{l+1}$, whence from (2.12) and (2.13),

$$\left|\sum_{j=n}^{l+n} a_{ln+j} y_k^j\right| \leqq M_5 + \frac{M_4 n |y_k|^n}{(\varrho-\varepsilon)^{(l+2)n}}. \tag{2.14}$$

On dividing through by $|y_k|^n$ in (2.14), we obtain

$$\left|\sum_{j=0}^{l} a_{n(l+1)+j} y_k^j\right| \leqq \frac{M_5}{|y_k|^n} + \frac{M_4 n}{(\varrho-\varepsilon)^{(l+2)n}}, \tag{2.15}$$

and so, from the definition of σ_1 in (2.5), there follows

$$\left|\sum_{j=0}^{l} a_{n(l+1)+j} y_k^j\right| \leqq \frac{M_5}{\sigma_1^n} + \frac{M_4 n}{(\varrho-\varepsilon)^{(l+2)n}}, \tag{2.16}$$

for all $n \geqq n_0(\varepsilon)$, all $1 \leqq k \leqq l+1$. If, for convenience, we set

$$\tau := \max\left\{\frac{1}{\sigma_1};\ \frac{1}{(\varrho-\varepsilon)^{l+2}}\right\}, \tag{2.17}$$

then it follows from (2.3) and (2.5) that

$$\tau < \frac{1}{\varrho^{l+1}}. \tag{2.18}$$

Next, we write a system of $(l+1)$ linear equations in the "unknowns" $a_{(l+1)n+j}$, i.e.,

$$\sum_{j=0}^{l} y_k^j a_{(l+1)n+j} =: f_{k,n}, \quad k = 1, 2, \ldots, l+1 \tag{2.19}$$

where, from (2.16) and (2.17),

$$|f_{k,n}| \leqq M_6 n \tau^n, \quad \forall n \geqq n_0(\varepsilon), \quad \forall 1 \leqq k \leqq l+1. \tag{2.20}$$

In matrix notation, we can write the system of equations (2.19) as

$$\begin{bmatrix} 1 & y_1 & \cdots & y_1^l \\ 1 & y_2 & \cdots & y_2^l \\ \vdots & & & \vdots \\ 1 & y_{l+1} & \cdots & y_{l+1}^l \end{bmatrix} \cdot \begin{bmatrix} a_{(l+1)n} \\ a_{(l+1)n+1} \\ \vdots \\ a_{(l+1)n+l} \end{bmatrix} = \begin{bmatrix} f_{1,n} \\ f_{2,n} \\ \vdots \\ f_{l+1,n} \end{bmatrix}. \tag{2.21}$$

The coefficient matrix, Δ, in (2.21) is a Vandermonde matrix, and, as the points $\{y_k\}_{k=1}^{l+1}$ are *distinct* by hypothesis, then Δ is nonsingular. Using Cramer's rule, it is easy to see from (2.20) and the fact that the $\{y_k\}_1^{l+1}$ are fixed distinct points, that

$$|a_{(l+1)n+j}| \leqq M_7 n\tau^n, \quad \forall n \geqq n_0(\varepsilon), \quad \forall 0 \leqq j \leqq l. \tag{2.22}$$

However, (2.22) implies that

$$\varlimsup_{n\to\infty} |a_n|^{1/n} \leqq \tau^{1/(l+1)} < \frac{1}{\varrho}, \tag{2.23}$$

the last inequality coming from (2.18). As this contradicts (2.2), then there can be at most l distinct points $\{\eta_k\}_{k=1}^{l}$ in $|z|>\varrho^{l+1}$ for which the sequence (1.8) is bounded, completing thefirst part of the proof.

To establish the second part of Theorem 1, let $w_l(z)$ be any monic polynomial of degree l with precisely l distinct zeros in the annulus $\varrho^{l+1}<|z|<\varrho^{l+2}$, i.e.,

$$w_l(z) = \prod_{k=1}^{l} (z-\eta_k) =: \sum_{j=0}^{l} \beta_j z^j, \tag{2.24}$$

where

$$\varrho^{l+1} < |\eta_k| < \varrho^{l+2} \quad \text{for} \quad k = 1, 2, \ldots, l. \tag{2.25}$$

Consider then the particular function

$$\hat{f}(z) := \frac{w_l(z)}{\varrho^{l+1}-z^{l+1}}. \tag{2.26}$$

Clearly, $\hat{f}\in A_\varrho$, and $\hat{f}$ has $l+1$ poles on $|z|=\varrho$. We now show that with these definitions, (1.9) of Theorem 1 is satisfied. From Theorem B, we know that

$$\lim_{n\to\infty} \left\{p_{n-1}(z;\hat{f}) - \sum_{j=0}^{l} P_{n-1,j}(z;\hat{f})\right\} = 0, \quad \forall z\in D_{\varrho^{l+2}}. \tag{2.27}$$

We claim that

$$\lim_{n\to\infty} P_{n-1,l}(\eta_k;\hat{f}) = 0, \quad \forall 1 \leqq k \leqq l. \tag{2.28}$$

To establish (2.28), write $\hat{f}(z) := \sum_{k=0}^{\infty} \hat{a}_k z^k$. It follows from (2.24) and (2.26) that

$$\hat{a}_{m(l+1)+j} = \frac{\beta_j}{\varrho^{(m+1)(l+1)}}, \quad \forall 0 \leqq j \leqq l, \quad \forall m \geqq 0. \tag{2.29}$$

Next, by definition,

$$P_{n-1,l}(z;\hat{f}) = \sum_{k=0}^{n-1} \hat{a}_{ln+k} z^k, \tag{2.30}$$

and we consider the case when n is a multiple of $(l+1)$, i.e., $n=(l+1)s$. On regrouping terms in (2.30) for such n, $P_{n-1,l}(z:\hat{f})$ can be expressed as

$$P_{s(l+1)-1,l}(z;\hat{f}) = \sum_{k=0}^{s-1} z^{k(l+1)} \sum_{j=0}^{l} \hat{a}_{(l+1)[sl+k]+j} z^j. \tag{2.31}$$

But, the inner sum of (2.31) can be seen from (2.29) and (2.24) to be

$$\sum_{j=0}^{l} \hat{a}_{(l+1)[sl+k]+j} z^j = \frac{w_l(z)}{\varrho^{(l+1)[sl+k+1]}}. \tag{2.32}$$

Since $w_l(\eta_k)=0$ by definition, it follows from (2.31) that

$$P_{s(l+1)-1,l}(\eta_k;\hat{f}) = 0, \quad \forall 1 \leqq k \leqq l, \quad \forall s \geqq 1. \tag{2.33}$$

Having just considered the case when n is a multiple of $(l+1)$, we now suppose that $n=s(l+1)+t$, where $1 \leqq t \leqq l$. On similarly regrouping the terms in (2.30) and using the fact that $w_l(\eta_k)=0$, it can be shown that

$$P_{s(l+1)+t-1,l}(\eta_k;\hat{f}) = \sum_{j=0}^{t-1} \hat{a}_{sl(l+1)+lt+j}\eta_k^j. \tag{2.34}$$

Since the $\{\eta_k\}_{k=1}^l$ are fixed, and t does not exceed l, then, as $|\hat{a}_n| \to 0$ as $n\to\infty$ from (2.29), we have from (2.33) and (2.34) that

$$\lim_{n\to\infty} P_{n-1,l}(\eta_k;\hat{f}) = 0, \quad \forall 1 \leqq k \leqq l, \tag{2.35}$$

as claimed in (2.28). Thus, with (2.27) and the first part of Theorem 1, the sequence

$$\left\{p_{n-1}(z;\hat{f}) - \sum_{j=0}^{l-1} P_{n-1,j}(z;\hat{f})\right\}_{n=1}^{\infty} \tag{2.36}$$

is convergent (to zero), only in the points $\{\eta_k\}_{k=1}^l$ and unbounded for all other points in $\{z\in \mathbf{C}\colon |z|>\varrho^{l+1}\}$.

Added in proof. (April 14, 1983) The second part of Theorem 1 remains valid if any l distinct points $\{\eta_k\}_{k=1}^l$ are arbitrarily chosen in $|z|>\varrho^{l+1}$, with a similar improvement holding for Theorem 2. This has been shown by the author and, more generally by T. Hermann, "Some remarks on an extension of a Theorem of Walsh", *J. Approx. Th.* (to appear).

References

[1] A. S. Cavaretta, Jr., A. Sharma and R. S. Varga, Interpolation in the roots of unity: an extension of a theorem of J. L. Walsh, *Resultate der Mathematik*, **3** (1981), 155—191.

[2] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, Amer. Math. Soc. Colloquium Publications Volume **XX** (Providence, Rhode Island, fifth edition, 1969).

(Received December 27, 1981)

CENTER FOR MATH. SERVICES
UNIVERSITY OF SOUTH FLORIDA
TAMPA, FL, 33620
U.S.A.

INSTITUTE FOR COMP. MATHEMATICS
KENT STATE UNIVERSITY
KENT, OH, 44242
U.S.A.

82-3336 — Szegedi Nyomda — F. v.: Dobó József igazgató

Periodicals of the Hungarian Academy of Sciences are obtainable at the following addresses: