p-Cyclic Matrices and the Symmetric Successive Overrelaxation Method

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ABSTRACT

In this paper, the new functional equation,

$$\left[\lambda-(1-\omega)^2\right]^p=\lambda[\lambda+1-\omega]^{p-2}(2-\omega)^2\omega^p\mu^p,$$

which connects the eigenvalues μ of a particular weakly cyclic (of index p) Jacobi matrix B to the eigenvalues λ of its associated symmetric successive overrelaxation (SSOR) matrix S_ω, is derived. This functional equation is then applied to the problem of determining bounds for the intervals of convergence and divergence of the SSOR iterative method for classes of H-matrices.

1. INTRODUCTION

The first purpose of this paper is to derive the new functional equation (see Section 2)

$$\left[\lambda-(1-\omega)^2\right]^p=\lambda[\lambda+1-\omega]^{p-2}(2-\omega)^2\omega^p\mu^p, \tag{1.1}$$

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form

$$B=\begin{bmatrix} 0 & B_{1,2} & 0 \\ 0 & 0 & B_{2,3} \\ B_{3,1} & 0 & 0 \end{bmatrix},$$

$$L=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B_{3,1} & 0 & 0 \end{bmatrix}, \tag{2.9}$$

$$U=\begin{bmatrix} 0 & B_{1,2} & 0 \\ 0 & 0 & B_{2,3} \\ 0 & 0 & 0 \end{bmatrix}.$$

A direct calculation with the above matrix L gives that

$$M(L)=\begin{bmatrix} (1-\omega)I & 0 & 0 \\ 0 & (1-\omega)I & 0 \\ \omega(2-\omega)B_{3,1} & 0 & (1-\omega)I \end{bmatrix}, \tag{2.10}$$

while a similar computation for the matrix U of (2.9) gives

$$M(U)=\begin{bmatrix} (1-\omega)I & \omega(2-\omega)B_{1,2} & \omega^2(2-\omega)B_{1,2}B_{2,3} \\ 0 & (1-\omega)I & \omega(2-\omega)B_{2,3} \\ 0 & 0 & (1-\omega)I \end{bmatrix}. \tag{2.11}$$

Hence, from (2.8), we have

$$\tilde{S}_\omega=(1-\omega)^2 I+T_\omega, \tag{2.12}$$

where

$$T_\omega:=\begin{bmatrix} 0 & \omega\sigma B_{1,2} & \omega^2\sigma B_{1,2}B_{2,3} \\ 0 & 0 & \omega\sigma B_{2,3} \\ \omega\sigma B_{3,1} & \omega^2(2-\omega)^2 B_{3,1}B_{1,2} & \omega^3(2-\omega)^2 B_{3,1}B_{1,2}B_{2,3} \end{bmatrix}, \tag{2.13}$$

and where

$$\sigma := (1-\omega)(2-\omega). \tag{2.14}$$

Clearly, τ is an eigenvalue of T_ω iff

$$\lambda := (1-\omega)^2 + \tau \tag{2.15}$$

is an eigenvalue of $\tilde{S}_\omega$.

We now derive a relationship between the eigenvalues of B and those of $\tilde{S}_\omega$. Assuming that τ is an eigenvalue of T_ω, there is a nonzero vector $[X_1, X_2, X_3]^T$ such that

$$T_\omega \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \tau \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \tag{2.16}$$

where the vector $[X_1, X_2, X_3]^T$ is partitioned conformally with respect to the partitioning of the matrices in (2.9). It follows from (2.13) that the subvectors X_1, X_2 and X_3 satisfy

$$\omega\sigma B_{1,2}X_2 + \omega^2\sigma B_{1,2}B_{2,3}X_3 = \tau X_1, \tag{2.17}$$

$$\omega\sigma B_{2,3}X_3 = \tau X_2, \tag{2.18}$$

$$\omega\sigma B_{3,1}X_1 + \omega^2(2-\omega)^2 B_{3,1}B_{1,2}X_2 + \omega^3(2-\omega)^2 B_{3,1}B_{1,2}B_{2,3}X_3 = \tau X_3. \tag{2.19}$$

Assuming $\tau \neq 0$, equation (2.18) gives that

$$X_2 = \frac{\omega\sigma}{\tau} B_{2,3}X_3, \tag{2.20}$$

while (2.17) and (2.20) together give that

$$X_1 = \left\{ \frac{\omega^2\sigma^2}{\tau^2} + \frac{\omega^2\sigma}{\tau} \right\} B_{1,2}B_{2,3}X_3. \tag{2.21}$$

On substituting (2.20) and (2.21) into (2.19) and on multiplying through by τ^2, there holds, after some simplifications using (2.14) and (2.15), that

$$\omega^3(2-\omega)^2\lambda[\lambda+1-\omega]B_{3,1}B_{1,2}B_{2,3}X_3=\left[\lambda-(1-\omega)^2\right]^3X_3. \quad (2.22)$$

As a consequence of (2.22), it follows, under the assumption that $\tau \neq 0$, that X_3 is not zero, for if, on the contrary, X_3 were zero, then the same would be true for the vectors X_1 and X_2 from (2.20) and (2.21), which contradicts the assumption that $[X_1, X_2, X_3]^T$ is an eigenvector of T.

Next, under the assumptions that $\tau \neq 0$ and that $0 < \omega < 2$, the coefficient of $B_{3,1}B_{1,2}B_{2,3}X_3$ in (2.22) can vanish only if $\lambda = 0$ or if $\lambda = \omega - 1$. If we assume first that $\lambda = 0$, it follows from (2.22) that the right side of (2.22) must vanish, and as $X_3 \neq 0$, then $\omega = 1$. But, from (2.15), then $\tau = 0$, a contradiction. Thus, $\lambda \neq 0$. If, in the remaining case, we assume that $\lambda = \omega - 1$, it again follows that $\omega = 1$, which from (2.15) again yields the contradiction that $\tau = 0$, whence $\lambda + 1 - \omega \neq 0$. Consequently, under the assumptions that $\tau \neq 0$ and that $0 < \omega < 2$, we can write (2.22) as

$$B_{3,1}B_{1,2}B_{2,3}X_3=\frac{\left[\lambda-(1-\omega)^2\right]^3}{\omega^3(2-\omega)^2\lambda[\lambda+1-\omega]}X_3, \quad (2.23)$$

i.e., X_3 is an eigenvector of the matrix $B_{3,1}B_{1,2}B_{2,3}$ with associated nonzero eigenvalue $[\lambda-(1-\omega)^2]^3/\omega^3(2-\omega)^2\lambda[\lambda+1-\omega]$.

We now make use of the weakly cyclic of index 3 character of the matrix B of (2.9). By direct computation,

$$B^3=\begin{bmatrix} B_{1,2}B_{2,3}B_{3,1} & 0 & 0 \\ 0 & B_{2,3}B_{3,1}B_{1,2} & 0 \\ 0 & 0 & B_{3,1}B_{1,2}B_{2,3} \end{bmatrix}, \quad (2.24)$$

where the three diagonal submatrices of B^3 are easily seen to have all the same nonzero eigenvalues. Moreover, it follows from a result of Romanovsky (cf. [10, p. 40]), that

$$B_{3,1}B_{1,2}B_{2,3}Y_3=\mu^3Y_3 \qquad (\mu\neq 0, \quad Y_3\neq 0) \quad (2.25)$$

iff μ is a nonzero eigenvalue of B. Thus, the combination of (2.23) and (2.25) gives that $\mu^3 = [\lambda-(1-\omega^2)]^3/\omega^3(2-\omega)^2\lambda[\lambda+1-\omega]$, whence

$$\left[\lambda-(1-\omega)^2\right]^3=\lambda[\lambda+1-\omega](2-\omega)^2\omega^3\mu^3, \quad (2.26)$$

where μ is an eigenvalue of B. In other words, we have shown that if λ is an eigenvalue of the SSOR matrix S_ω for which $\lambda-(1-\omega)^2\neq 0$, if $0<\omega<2$, and if μ satisfies (2.26), then μ is an eigenvalue of the block Jacobi matrix B of (2.9).

Conversely, we claim that if μ is an eigenvalue of B, if $0<\omega<2$, and if $\hat{\lambda}$ satisfies (2.26) with $\hat{\lambda}\neq(1-\omega)^2$, then $\hat{\lambda}$ is an eigenvalue of S_ω. To establish this, it is evident from the hypotheses that μ cannot be zero. As before, μ is a nonzero eigenvalue of B iff $B_{3,1}B_{1,2}B_{2,3}Y_3=\mu^3 Y_3$ for some $Y_3\neq 0$. With $\hat{\tau}:=\hat{\lambda}-(1-\omega)^2$, where $\hat{\lambda}$ is the solution of (2.26), then by hypothesis, $\hat{\tau}\neq 0$. Hence, we define the vectors Y_2 and Y_3 by means of

$$Y_2:=\frac{\omega\sigma}{\hat{\tau}}B_{2,3}Y_3,\qquad Y_1:=\left\{\frac{\omega^2\sigma^2}{\hat{\tau}^2}+\frac{\omega^2\sigma}{\hat{\tau}}\right\}B_{1,2}B_{2,3}Y_3. \tag{2.27}$$

One can then verify from (2.17)–(2.19) that $T_\omega[Y_1,Y_2,Y_3]^T=\hat{\tau}[Y_1,Y_2,Y_3)^T$, so that $\hat{\lambda}$ is an eigenvalue of $\tilde{S}_\omega$, as claimed.

Actually, the above technique of proof for the case $p=3$ can be extended to the case of *any* weakly cyclic matrix B of index $p\geqslant 2$, as considered in (2.3). Leaving the details of the extension to the reader, we simply state our generalization of (2.26) as

THEOREM 1. *Given the matrix A of* (2.2) *with nonsingular square diagonal submatrices $A_{i,i}$, $1\leqslant i\leqslant p$, let B of* (2.3) *be its associated weakly cyclic (of index p) block Jacobi matrix, and assume $0<\omega<2$. If λ is an eigenvalue of S_ω for which $\lambda\neq(1-\omega)^2$, and if μ satisfies*

$$\left[\lambda-(1-\omega)^2\right]^p=\lambda[\lambda+1-\omega]^{p-2}(2-\omega)^2\omega^p\mu^p, \tag{2.28}$$

then μ is an eigenvalue of the block Jacobi matrix B of (2.3). *Conversely, if μ is an eigenvalue of B and if $\hat{\lambda}$ satisfies* (2.28) *with $\hat{\lambda}\neq(1-\omega)^2$, then $\hat{\lambda}$ is an eigenvalue of S_ω.*

The functional equation (2.28), relating the eigenvalues μ of the block Jacobi matrix B of (2.3) with the eigenvalues λ of the SSOR iteration matrix S_ω, has very much the *flavor* of Young's equation (cf. [12, p. 142])

$$(\lambda+\omega-1)^2=\lambda\omega^2\mu^2, \tag{2.29}$$

which similarly relates the eigenvalues of a consistently ordered weakly cyclic (of index 2) Jacobi matrix B with the eigenvalues λ of the associated SOR iteration matrix L_ω, as well as of Varga's extension of Young's equation (cf. [10, p. 106])

$$(\lambda+\omega-1)^p=\lambda^{p-1}\omega^p\mu^p, \tag{2.30}$$

which relates the eigenvalues of a consistently ordered weakly cyclic (of index $p\geqslant 2$) Jacobi matrix B with the eigenvalues of the associated SOR iteration matrix L_ω. It is interesting to remark that the textbook proofs *usually* used to establish (2.29) or (2.30) involve first results on determinantal invariance (cf. [10, p. 102], [12, p. 141]), a step which has *not* been directly used in the technique of proof of our Theorem 1. Now, this lack of such a determinantal invariance in the SSOR case may account for the fact that such extensions (relating the eigenvalues of the SSOR iteration matrix S_ω with the eigenvalues of the Jacobi matrix B when it is weakly cyclic of index p) have not appeared earlier in the SSOR literature. Thus, Theorem 1 appears to fill this gap in the SSOR literature.

Some easy consequences of Theorem 1 are worth mentioning. If $p=2$, then (2.28) reduces to

$$\left[\lambda-(1-\omega)^2\right]^2=\lambda(2-\omega)^2\omega^2\mu^2, \tag{2.31}$$

which was obtained earlier by D'Sylva and Miles [2]. In particular, on setting $\hat{\omega}:=\omega(2-\omega)$, the above takes the more familiar form

$$[\lambda+\hat{\omega}-1]^2=\lambda\hat{\omega}^2\mu^2, \tag{2.32}$$

which is Young's equation (2.29). From this, we recover earlier results of Niethammer [8] and Lynn [4] concerning the SSOR method in the case that the Jacobi matrix B is weakly cyclic of index 2.

We further remark that our explicit technique for deriving (2.28) of Theorem 1 can also be used to directly derive Varga's functional relation

$$(\lambda+\omega-1)^p=\lambda^{p-1}\omega^p\mu^p \qquad (p\geqslant 2) \tag{2.33}$$

between the eigenvalues μ of a consistently ordered weakly cyclic (of index p) Jacobi matrix B and the eigenvalues λ of the associated SOR iteration matrix L_ω, *without* using results on determinantal invariance. In fact, this technique allows one to similarly derive the analogous results (cf. Kjellberg [3], Nichols and Fox [7], and Varga [10, Example 2, p. 109])

$$(\lambda+\omega-1)^p=\lambda^k\omega^p\mu^p \qquad (k=1,2,\ldots,p-1;\quad p\geqslant 2) \tag{2.34}$$

between the eigenvalues μ of a weakly p-cyclic Jacobi matrix B and the eigenvalues λ of the associated SOR matrix L_ω, where the *effect* of ordering is seen in the exponent of λ in (2.34).

We finally remark that functional equations, similar to (2.28), can also be derived for permutation transformations applied to the weakly cyclic of index p Jacobi matrix B of (2.3). For example, if in place of B of (2.9), we consider its "permuted" matrix

$$\tilde{B} = \begin{bmatrix} 0 & 0 & \tilde{B}_{1,3} \\ \tilde{B}_{2,1} & 0 & 0 \\ 0 & \tilde{B}_{3,2} & 0 \end{bmatrix}, \tag{2.35}$$

which is also weakly cyclic of index 3, then it can be verified that the functional equation (2.28) of Theorem 1 is *unchanged*. On the other hand, if we consider the weakly cyclic (of index 4) matrix

$$\hat{B} = \begin{bmatrix} 0 & 0 & \hat{B}_{1,3} & 0 \\ 0 & 0 & 0 & \hat{B}_{2,4} \\ 0 & \hat{B}_{3,2} & 0 & 0 \\ \hat{B}_{4,1} & 0 & 0 & 0 \end{bmatrix}, \tag{2.36}$$

which is a permutation of that matrix considered in (2.3) for $p = 4$, one obtains instead the functional equation

$$\left[\lambda - (1-\omega)^2\right]^4 = \lambda^2(2-\omega)^4\omega^4\mu^4, \tag{2.37}$$

relating the eigenvalues μ of the matrix $\hat{B}$ with the eigenvalues λ of the associated SSOR matrix S_ω.

3. APPLICATIONS TO H-MATRICES

In this section, we use the result of Theorem 1 to deduce new upper bounds for the domain of convergence of the SSOR iterative method when applied to H-matrices. We first begin with some necessary notation.

Let $\mathbb{C}^{n,n}$ ($\mathbb{R}^{n,n}$) denote the set of all $n \times n$ matrices $A = [a_{i,j}]$ with complex (real) entries. For each $A = [a_{i,j}]$ in $\mathbb{C}^{n,n}$, let $\mathfrak{M}(A) := [\alpha_{i,j}]$ in

$\mathbb{R}^{n,n}$ be the *comparison matrix* for A, defined by

$$\alpha_{i,i} := |a_{i,i}|, \quad 1 \leqslant i \leqslant n; \quad \alpha_{i,j} := -|a_{i,j}|, \quad i \neq j; \qquad 1 \leqslant i, \ j \leqslant n. \quad (3.1)$$

Further, for any $A = [a_{i,j}] \in \mathbb{C}^{n,n}$, we set

$$\Omega(A) := \left\{ B = \left[b_{i,j} \right] \in \mathbb{C}^{n,n} : |b_{i,j}| = |a_{i,j}| \text{ for all } 1 \leqslant i, \ j \leqslant n \right\}. \quad (3.2)$$

We call $\Omega(A)$ the *equimodular set of matrices* associated with A. Note that both A and $\mathfrak{M}(A)$ are in $\Omega(A)$.

Next, let $\mathbb{C}_{\pi}^{n,n}$ denote the subset of matrices in $\mathbb{C}^{n,n}$ having all diagonal entries nonzero. Then for each $A = [a_{i,j}] \in \mathbb{C}_{\pi}^{n,n}$, we can decompose each $B = [b_{i,j}] \in \Omega(A)$ into the sum

$$B = D(B) - L(B) - U(B) \quad (3.3)$$

where $D(B) := \operatorname{diag}[b_{1,1}, b_{2,2}, \ldots, b_{n,n}]$, and where $L(B)$ and $U(B)$ are respectively strictly lower and strictly upper triangular matrices. We then define

$$J(B) := (D(B))^{-1}\{L(B) + U(B)\} \qquad [\text{for all} \quad B \in \Omega(A)] \quad (3.4)$$

to be the associated (point) *Jacobi matrix* for B.

Next, any matrix $B = [b_{i,j}] \in \mathbb{R}^{n,n}$ with $b_{i,j} \leqslant 0$ for all $i \neq j$, $1 \leqslant i, \ j \leqslant n$, can be expressed as

$$B = \tau I - C, \quad (3.5)$$

where $\tau := \max_{1 \leqslant i \leqslant n} b_{i,i}$, and where $C = [c_{i,j}] \in \mathbb{R}^{n,n}$, having nonnegative entries, is defined by

$$c_{i,i} = \tau - b_{i,i} \geqslant 0; \quad c_{i,j} = -b_{i,j} \geqslant 0, \ i \neq j; \qquad 1 \leqslant i, j \leqslant n. \quad (3.6)$$

Following Ostrowski [9], such a matrix is said to be a *nonsingular M-matrix* if $\tau > \rho(C)$, where $\rho(C) := \max\{|\lambda| : \det(\lambda I - C) = 0\}$ is the spectral radius of C, and any matrix $A \in \mathbb{C}^{n,n}$ for which $\mathfrak{M}(A)$ is a nonsingular M-matrix is similarly called a nonsingular *H-matrix*. Finally if $B = [b_{i,j}] \in \mathbb{C}^{n,n}$, then $|B| := [|b_{i,j}|] \in \mathbb{R}^{n,n}$.

Now, from the work of Alefeld and Varga [1], Neumann [5], and Varga [11], it can be shown that, given any $A \in \mathbb{C}_{\pi}^{n,n}$, $n \geqslant 2$, the following are equivalent:

(i) A is a nonsingular H-matrix;

(ii) for any $B \in \Omega(A)$, $\rho(J(B) \leqslant \rho(|J(B)|) = \rho(J(\mathfrak{M}(A))) < 1$;

(iii) for each $B \in \Omega(A)$ and for each ω in the interval $0 < \omega < 2/[1+\rho(|J(B)|)]$, the associated SOR iteration matrix $L_\omega(B)$ satisfies

$$\rho(L_\omega(B)) \leqslant \omega\rho(|J(B)|) + |1-\omega| < 1, \tag{3.7}$$

i.e., $L_\omega(B)$ is convergent;

(iv) for each $B \in \Omega(A)$ and for each ω in the interval $0 < \omega < 2/[1+\rho(|J(B)|)]$, the associated SSOR iteration matrix $S_\omega(B)$ satisfies

$$\rho(S_\omega(B)) \leqslant \omega\rho(|J(B)|) + |1-\omega| < 1, \tag{3.8}$$

i.e., $S_\omega(B)$ is convergent.

To discuss the sharpness of the first inequality in (3.7), set

$$\mathscr{H}_\nu := \{A \in \mathbb{C}^{n,n},\ n \text{ arbitrary}: A \text{ is an } H\text{-matrix with } \rho(|J(A)|) = \nu\} \quad \text{for each } \nu \in [0,1). \tag{3.9}$$

With this notation (3.7) can be expressed as

$$\rho(L_\omega(B)) \leqslant \omega\nu + |1-\omega|, \quad \text{for all} \quad 0 < \omega < \frac{2}{1+\nu}, \quad \text{all} \quad B \in \mathscr{H}_\nu, \tag{3.10}$$

from which it is evident that

$$\sup\{\rho(L_\omega(B)) : B \in \mathscr{H}_\nu\} \leqslant \omega\nu + |1-\omega| \qquad \text{for all} \quad 0 < \omega < \frac{2}{1+\nu}. \tag{3.11}$$

It was shown by Neumann and Varga [6] that equality holds in (3.11), i.e.,

$$\sup\{\rho(L_\omega(B)) : B \in \mathscr{H}_\nu\} = \nu\omega + |1-\omega| \qquad \text{for all} \quad 0 \leqslant \omega \leqslant \frac{2}{1+\nu}, \tag{3.12}$$

so that the inequality of (3.10) cannot be improved (relative to the set $\mathscr{H}_\nu$). In this sense, the first inequality in (3.10) is sharp.

It is natural to similarly ask if the first inequality of (3.8) in (iv) is sharp, but this remains an open question. A simpler question is if the interval of convergence in ω, namely $(0,2/[1+\rho(|J(A)|)])$, is sharp in (iv) for each $B \in \Omega(A)$, where A is a nonsingular H-matrix. Since $S_0 = I$ from (2.5), it is trivial that convergence cannot hold at the left endpoint of $(0,2/[1+\rho(|J(A)|)])$. As for the right endpoint, Neumann [5] has recently shown, somewhat surprisingly, that

$$\rho(S_{\omega'}(A)) < 1 \qquad \left(\omega' := \frac{2}{1+\rho(|J(A)|)}\right)$$

for any nonsingular H-matrix A, unless $\rho(|J(A)|) = 0$. This suggests that the upper bound for ω, namely $2/[1+\rho(|J(A)|)]$, in (iv) might be improved.

To contribute to the above question, we now apply Theorem 1 to deduce new upper bounds for the domain of convergence of the SSOR iterative method when applied to H-matrices. Specifically, consider the $n \times n$ matrix $A = I - B$, where B is given by

$$B = \begin{bmatrix} 0 & \alpha & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \alpha \\ \beta & & & & 0 \end{bmatrix} \qquad (n \geqslant 2). \tag{3.13}$$

Clearly, B is a weakly cyclic matrix of index n, with $\nu^n := \alpha^{n-1}\beta$ an eigenvalue of B^n. Thus, (2.28) of Theorem 1 becomes in this case

$$\left[\lambda - (1-\omega)^2\right]^n - \lambda[\lambda + 1 - \omega]^{n-2}(2-\omega)^2\omega^n\nu^n = 0. \tag{3.14}$$

On setting $\lambda = -s$, and on choosing any complex number ν such that $-\nu^n = |\nu|^n$ and such that $|\nu|^n < 1$, the above equation can be expressed as

$$G_n(s;\omega,|\nu|) = 0, \tag{3.15}$$

where

$$G_n(s;\omega,\nu) := \left[s + (\omega-1)^2\right]^n - s[s+\omega-1]^{n-2}(2-\omega)^2\omega^n\nu^n. \tag{3.16}$$

In particular, for $s=1$ and for any $0<\omega<2$, we have

$$G_n(1;\omega,\nu)=[\omega^2-2\omega+2]^n-\left(\frac{2-\omega}{\omega}\right)^2[\nu\omega^2]^n. \tag{3.17}$$

It is evident that if

$$\nu\omega^2>\omega^2-2\omega+2 \qquad (\text{for} \quad 0<\omega<2), \tag{3.18}$$

then

$$G_n(1;\omega,\nu)<0 \qquad \text{for each } n \text{ sufficiently large.} \tag{3.19}$$

On setting

$$\hat{\omega}(\nu):=\frac{2}{1+\sqrt{2\nu-1}} \qquad \text{for each } \nu \text{ with } \tfrac{1}{2}<\nu<1, \tag{3.20}$$

so that $\hat{\omega}(\nu)>1$, one verifies directly that (3.18) holds, for $0<\omega<2$, precisely when

$$\hat{\omega}(\nu)<\omega<2. \tag{3.21}$$

On the other hand, (3.16) shows that G_n is a real polynomial in s with leading coefficient positive, so that

$$G_n(s;\omega,\nu)>0 \qquad \text{for all } s>0 \text{ sufficiently large.} \tag{3.22}$$

Thus, for each ω satisfying (3.21) and for each n sufficiently large, we see from (3.19) and (3.22) that there is an $s_n(\omega;\nu)$, satisfying

$$1<s_n(\omega;\nu), \tag{3.23}$$

for which $G_n(s_n(\omega;\nu);\omega,\nu)=0$. Equivalently, there is a $\hat{\lambda}_n:=-s_n(\omega;\nu)$ which satisfies (3.14). Now, as

$$\hat{\lambda}_n=-s_n(\omega;\nu)<-1 \tag{3.24}$$

from (3.21), then $\hat{\lambda}_n\neq(1-\omega)^2$ for each n. If $S_\omega^{(n)}$ denotes the SSOR matrix associated with the Jacobi matrix B of (3.13) (with $-\alpha^{n-1}\beta=\nu^n$), then on

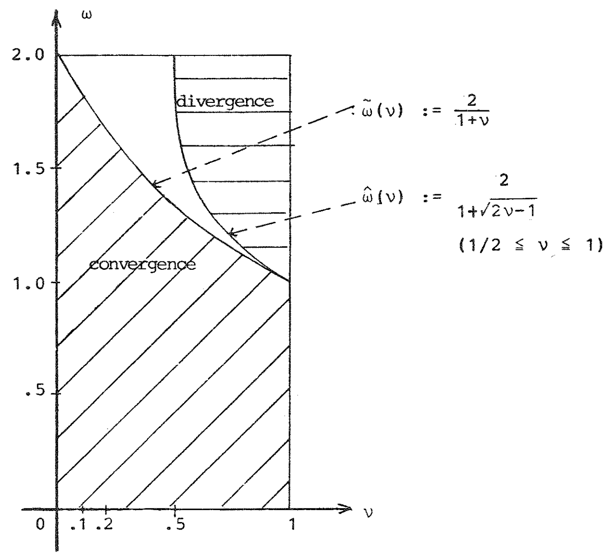

FIG. 1.

applying the last part of Theorem 1, $\hat{\lambda}_n$ must be an eigenvalue of $S_\omega^{(n)}$ for each ω satisfying (3.21). But as $|\hat{\lambda}_n| > 1$ from (3.24), then

$$\rho\left(S_\omega^{(n)}\right) > 1 \tag{3.25}$$

for each ω satisfying (3.21), for each n sufficiently large, so that the associated $n \times n$ SSOR matrix $S_\omega^{(n)}$ is necessarily divergent. Obviously, as the $n \times n$ matrix $A_n(\nu)$, corresponding to the choice of the matrix B in (3.13) (with $-\alpha^{n-1}\beta = \nu^n$), is by construction a matrix in the set $\mathscr{H}_\nu$, we have established the result of

THEOREM 2. *For each ν with $\frac{1}{2} < \nu < 1$, and for each ω satisfying $\hat{\omega}(\nu) < \omega < 2$ (cf. (3.20)), there holds*

$$\sup\{\rho(S_\omega(B)) : B \in \mathscr{H}_\nu\} > 1. \tag{3.26}$$

To illustrate the result of Theorem 2, we have drawn Figure 1, indicating the open region where divergence may take place [i.e., $2 > \omega > \hat{\omega}(\nu)$], as well as the open region where convergence takes place [i.e., $0 < \omega < 2/(1+\nu)$], within the class of matrices $\mathscr{H}_\nu$.

It is an open question whether convergence, divergence, or both occur in the unshaded region of Figure 1.

As directly suggested by Figure 1, we remark that the two curves there [i.e., $\tilde{\omega}(\nu)$ and $\hat{\omega}(\nu)$] can be shown to be tangent to one another at $\nu = 1$. In particular, this implies that the *interval of uncertainty* in ω, corresponding to the unshaded region in Figure 1, is *small* for ν close to unity. For example, when $\nu = 0.8$, we have

(1) *convergence*, for each $0 < \omega \leqslant \tilde{\omega}(0.8) \doteq 1.111$, of the associated (point) SSOR matrix associated with *any* matrix in $\mathscr{H}_{0.8}$;

(2) *divergence*, for each $1.127 \doteq \hat{\omega}(0.8) < \omega < 2$, of an associated (point) SSOR matrix associated with *some* matrix in $\mathscr{H}_{0.8}$.

Note added in proof: This open question is essentially answered in A. Neumaier and R. S. Varga, "Exact convergence and divergence domains for the symmetric successive overrelaxation (SSOR) iterative method applied to H-matrices," this issue, pp. 261.

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