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A SURVEY OF RECENT RESULTS ON ITERATIVE METHODS FOR SOLVING LARGE SPARSE LINEAR SYSTEMS

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I. INTRODUCTION

The most spectacular advances, in my opinion, which have taken place in recent years in the iterative solution of large sparse systems of linear equations, have come in the following areas: i) preconditioned conjugate gradient methods; ii) multigrid methods for solving elliptic difference equations; iii) efficient adaptation of basic iterative methods to special computers, such as vector computers; and iv) robust software. It is not surprising that over fifty percent of the papers in this Proceedings deal with these same items.

Because these above items are well-represented in these Proceedings, I have chosen to concentrate here specifically on the recent development in algebraic tools, which, by virtue of their reoccurring use in these above items, furnishes a link between current technology and future advances. As an excellent example of an algebraic tool which opened new doors in the area of matrix iterative analysis, and which is still widely used today, it would be difficult to find one better than Young's famous functional equation (cf. [26] and [27, p. 142]):

$$(\lambda + \omega - 1)^2 = \lambda\omega^2\mu^2 , \qquad (1.1)$$

coupling the eigenvalues λ of the successive overrelaxation (SOR) iteration matrix with the eigenvalues μ of the associated Jacobi matrix, in the consistently ordered weakly-cyclic of index 2 case. It is interesting that conformal

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ISBN 0-12-100560-7

mapping ideas were basic both for the analysis of (1.1), as well as for the analysis of its extension

$$(\lambda + \omega - 1)^p = \lambda^{p-1}\omega^p\mu^p \tag{1.2}$$

in the consistently ordered weakly-cyclic of index p ($p \geq 2$) case (cf. [20] and [22, p. 106]). While it is true that (1.1) and (1.2) can be derived by purely algebraic methods, it is curious that all of the well-recognized textbooks covering this theory have suppressed such conformal mapping ideas in favor of a purely algebraic approach to this problem (cf. [2],[22],[27]).

On the other hand, the recent analyses for example of Niethammer and Varga [16] and Eiermann and Niethammer [5] make new use of ideas from summability theory and conformal mapping theory, to study optimized iterative methods. (The latter work also brings into play the well-known concepts of "maximal convergence" and "uniformly distributed nodes" from complex function theory and approximation theory, concepts which should prove to be invaluable as research tools in the future of the area.) Because these latter papers are long, and will appear before the publication of the Proceedings of this Conference, I will not survey this topic here.

What, then, are some common algebraic tools used today in research in iterative analysis? Obviously, diagonal dominance arguments abound in the literature, but it is perhaps not so well-known that such arguments have a basis in the theory of M- and H-matrices, as introduced by Ostrowski [17] in 1937. It is also true that concepts arising from the Perron-Frobenius theory of nonnegative matrices, such as weakly-cyclic matrices of index p (≥ 2), are in frequent use today. Because of this, I have chosen to focus on closely-related algebraic methods arising in the recent analysis of the symmetric successive overrelaxation (SSOR) iterative method, in the iterative solution of sparse least-squares problems, and in extensions in the theory of regular splittings.

II. A NEW IDENTITY FOR THE SSOR METHOD

Today, a popular preconditioning method used in conjunction with the conjugate gradient method, is one or more sweeps of the SSOR method. This new use of the SSOR iterative method has, interestingly enough, sparked recent interest into the general theory of this iterative method. The purpose of this section is to review some new developments in the basic theory for the SSOR method, in the weakly-cyclic case (of index $p \geq 2$), which parallel those for the SOR method.

To begin with, consider the matrix equation

$$A\underline{x} = \underline{k} , \tag{2.1}$$

where it is assumed that the $n \times n$ matrix has the partitioned form

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & 0 & \cdots & 0 & 0 \\ 0 & A_{2,2} & A_{2,3} & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & A_{p-1,p-1} & A_{p-1,p} \\ A_{p,1} & 0 & 0 & \cdots & 0 & A_{p,p} \end{bmatrix} . \tag{2.2}$$

Suppose also that $p \geq 2$ and that each diagonal submatrix $A_{i,i}$ is square and nonsingular. On setting $D:= \text{diag}[A_{1,1};A_{2,2};\ldots;A_{p,p}]$, the associated block-Jacobi matrix $J:=I - D^{-1}A$ has the form

$$J = \begin{bmatrix} 0 & B_{1,2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & B_{2,3} & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & B_{p-1,p} \\ B_{p,1} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} , \tag{2.3}$$

so that J is weakly-cyclic of index p (cf. [22, p. 39]). Thus, on expressing the matrix J as the sum of a strictly lower and strictly upper triangular matrix, i.e.,

$$J = L + U , \tag{2.4}$$

the associated SSOR iteration matrix S_ω is defined as usual (cf. [27, p. 461]) as

$$S_\omega := (I-\omega U)^{-1}[(1-\omega)I+\omega L]\cdot(I-\omega L)^{-1}[(1-\omega)I+\omega U] \quad , \tag{2.5}$$

while the associated SOR (successive overrelaxation) iterative matrix L_ω is similarly defined (cf. [27, p. 73]) as

$$L_\omega := (I-\omega L)^{-1}[(1-\omega)I + \omega U] \quad . \tag{2.6}$$

In both of the above matrices, ω is the relaxation parameter.

The new identity coupling the eigenvalues λ of the SSOR iteration matrix S_ω to the eigenvalues μ of the block-Jacobi matrix J of (2.3) is contained in

Theorem 2.1 ([24]). Given the matrix A of (2.2) with nonsingular square diagonal submatrices $A_{i,i}$, $1 \le i \le p$, let J of (2.3) be its associated weakly-cyclic of index p block Jacobi matrix, and assume that $0 < \omega < 2$. If λ is any eigenvalue of the associated SSOR iteration matrix S_ω (of (2.5)) for which $\lambda \neq (1-\omega)^2$ and if μ satisfies

$$[\lambda-(1-\omega)^2]^p = \lambda[\lambda+1-\omega]^{p-2}(2-\omega)^2\omega^p\mu^p \quad , \tag{2.7}$$

then μ is an eigenvalue of the block-Jacobi matrix J. Conversely, if μ is any eigenvalue of J and if $\hat{\lambda}$ satisfies (2.7) with $\hat{\lambda} \neq (1-\omega)^2$, then $\hat{\lambda}$ is an eigenvalue of S_ω.

It is interesting that the tools used to obtain these results stem from the Perron-Frobenius theory of nonnegative matrices, the canonical form of a weakly-cyclic of index p matrix, and the Frobenius-Romanovsky Theorem on the eigenvalues of a weakly-cyclic of index p matrix (cf. [22, p. 39, 40]).

We first remark that the functional equation (2.7), relating the eigenvalues μ of the block-Jacobi matrix J of (2.3) with the eigenvalues λ of the SSOR iteration matrix S_ω of (2.5), has very much the flavor of both Young's functional equation (1.1) as well as Varga's extension (1.2) of it (for $p > 2$) for the SOR iterative method. It is evident, from the tools used in the proof given in [24], that Theorem 2.1 could have been derived twenty years earlier. Why wasn't it? One interpretation for this late discovery of Theorem 2.1 may be the following. The usual textbook proof of (1.1)-(1.2) involve a preliminary result on a certain determinantal invariance (cf. [22, p. 102] and [27, p. 141]), a step which is not used in [24] in deriving Theorem 2.1. It is plausible that this lack of a similar determinantal invariance in the SSOR case may have been responsible for this delay in establishing Theorem 2.1.

It should also be mentioned that the special case p = 2 of (2.7) of Theorem 2.1 was derived earlier by D'Sylva and Miles [19]. In particular, on setting $\hat{\omega} := \omega(2-\omega)$, the special case p = 2 of (2.7) becomes

$$[\lambda + \hat{\omega} - 1]^2 = \lambda\hat{\omega}^2\mu^2 , \tag{2.8}$$

which reduces to Young's equation (1.1). From this, one easily recovers earlier results of Niethammer [14] and Lynn [9] concerning the SSOR method in the case when the Jacobi matrix is a weakly-cyclic of index 2 matrix.

In the case of applying the SOR iterative method to a block-Jacobi matrix which is weakly cyclic of index p, the effect of reorderings (or permutations) of the associated unknowns appears as a change in the exponent k of λ^k in the functional equation of (1.2):

$$(\lambda+\omega-1)^p = \lambda^k\omega^p\mu^p , \quad (k = 1,2,\ldots,p-1;\ p \geq 2), \tag{2.9}$$

(cf. [22, p. 109, Exercise 2], and Nickels and Fox [13]). It is an open question how such reorderings of unknowns similarly affect the functional equation of (2.7) of Theorem 2.1, for the SSOR iterative method.

III. CONVERGENCE AND DIVERGENCE DOMAINS FOR SSOR, APPLIED TO H-MATRICES

As mentioned in §II, the popularity of the SSOR iterative method as a preconditioner, in conjunction with the conjugate gradient method, has revived interest in the general theory of the SSOR iterative method. The purpose of this section is to review new developments in this theory, as it pertains to H-matrices. As we shall see, these new results parallel similar results for the SOR method.

To begin with, a certain amount of notation is necessary. Let $\mathbb{C}^{m,n}$ ($\mathbb{R}^{m,n}$) denote the set of all mxn matrices $A = [a_{i,j}]$ having complex (real) entries. For each $A = [a_{i,j}]$ in $\mathbb{C}^{n,n}$, the comparison matrix for A, $M(A) = [\alpha_{i,j}]$, has its entries $\alpha_{i,j}$ defined by

$$\alpha_{i,i} := |a_{i,i}|,\ 1 \leq i \leq n; \quad \alpha_{i,j} := -|a_{i,j}|,\ i \neq j, \quad 1 \leq i,j \leq n, \tag{3.1}$$

so that $M(A)$ is in $\mathbb{R}^{n,n}$. Further, for any $A = [a_{i,j}]$ in $\mathbb{C}^{n,n}$, set

$$\Omega(A) := \{B = [b_{i,j}] \in \mathbb{C}^{n,n} : |b_{i,j}| = a_{i,j}, \text{ for all } 1 \leq i,j \leq n\}. \tag{3.2}$$

The set $\Omega(A)$ is called the equimodular set of matrices associated with A. Note that both A and $M(A)$ are in $\Omega(A)$.

Next, let $\mathbb{C}_\pi^{n,n}$ denote the subset of matrices in $\mathbb{C}^{n,n}$ having all diagonal entries nonzero. For each $B = [b_{i,j}]$ in $\mathbb{C}_\pi^{n,n}$, we can express B as the sum

$$B = D(B)\{I - L(B) - U(B)\}, \tag{3.3}$$

where $D(B) := \text{diag}[b_{1,1}, b_{2,2}, \ldots, b_{n,n}]$ is nonsingular, and where $L(B)$ and $U(B)$ are respectively strictly lower and strictly upper triangular matrices. Then,

$$J(B) = L(B) + U(B) \tag{3.4}$$

as before in (2.4), defines the associated (point) Jacobi matrix for B. From a classic paper of Ostrowski [17], we know that any real nonsingular matrix $A = [a_{i,j}]$, with $a_{i,j} \leq 0$ for all $i \neq j$ ($1 \leq i,j \leq n$) and with A^{-1} having only nonnegative entries, is a nonsingular M-matrix, and that any matrix A in $\mathbb{C}^{n,n}$, for which its comparison matrix $M(A)$ is a nonsingular M-matrix, is by definition a non-singular H-matrix. Finally, if $B = [b_{i,j}]$ is in $\mathbb{C}^{m,n}$, then the matrix $|B|$ is defined as usual as $|B| = [|b_{i,j}|]$, and $\rho(C) := \max\{|\lambda| : \det(\lambda I - C) = 0\}$ denotes the spectral radius of any nxn matrix C.

From the works of Alefeld and Varga [1] and Varga [23], we also know that, given any A in $\mathbb{C}_\pi^{n,n}$, $n \geq 2$, the following are equivalent:

i) A is a nonsingular H-matrix;

ii) for any $B \in \Omega(A)$, $\rho(J(B)) \leq \rho(|J(B)|) = \rho(J(M(A))) < 1$;

iii) for each $B \in \Omega(A)$ and for each ω satisfying $0 < \omega < 2/[1+\rho(|J(B)|)]$, the associated SOR iteration matrix $L_\omega(B)$ for B of (2.6), satisfies

$$\rho(L_\omega(B)) \leq |1-\omega| + \omega\rho(|J(B)|) < 1 \; ; \tag{3.5}$$

iv) for each $B \in \Omega(A)$ and for each ω satisfying $0 < \omega < 2/[1+\rho(|J(B)|)]$, the associated SSOR iteration matrix $S_\omega(B)$ for B of (2.5) satisfies

$$\rho(S_\omega(B)) < 1. \tag{3.6}$$

We remark that the inequality of (3.5) originates in the works of Kahan [7] and Kulisch [8].

To discuss the sharpness of the first inequality of (3.5), we set

$$H_\nu := \{A \in \mathbb{C}^{n,n},\ n \text{ arbitrary: A is a nonsingular H-matrix with } \rho(|J(A)|) = \nu\}, \text{ where } 0 \leq \nu < 1. \tag{3.7}$$

Thus, with the definition of (3.7), it directly follows from (3.5) that

$$\sup\{\rho(L_\omega(A)) : A \in H_\nu\} \leq |1-\omega| + \omega\nu \quad (\text{for all } 0 < \omega < 2/(1+\nu)), \tag{3.8}$$

and it is natural to ask if equality holds in (3.8) for each ω with $0 < \omega < 2/(1+\nu)$. This was answered affirmatively in Neumann and Varga [12]:

$$\sup\{\rho(L_\omega(A)) : A \in H_\nu\} = |1-\omega| + \omega\nu \quad (\text{for all } 0 < \omega < 2/(1+\nu)). \tag{3.9}$$

We next ask in what sense iv) above is sharp for the SSOR iterative method. Since the SSOR iteration matrix $S_\omega(B)$ reduces to the identity matrix when $\omega = 0$ (cf. (2.5)), it is obvious that convergence of the SSOR matrix $S_\omega(B)$ cannot hold in iv) at the left endpoint of the interval $(0,2/[1+\rho(|J(A)|)]$, for any B in $\Omega(A)$. As for the right endpoint of this interval in iv), Neumann [11] has recently shown that

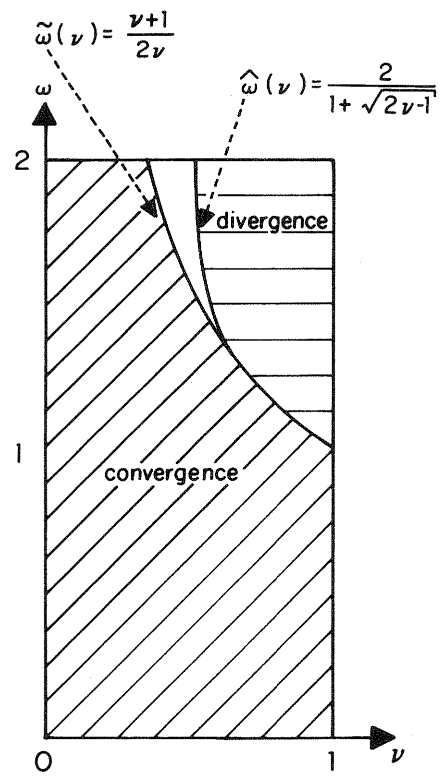

FIGURE 1.

IV. BLOCK ITERATIVE METHODS FOR SPARSE LEAST-SQUARES PROBLEMS

There has been much recent interest by numerical analysts in methods for accurately computing the least-squares solutions of very large sparse overdetermined linear systems of equations. In geodetical network problems, for example, such overdetermined systems take the form

$$A\underline{x} \doteq \underline{b} \,, \tag{4.1}$$

where A is a given real $m \times n$ matrix with $m \geq n$, and $\underline{b}$ is a given real vector with m components, (written $\underline{b} \in \mathbb{R}^m$). Usually, m is very much larger than n. Assuming that A has full column rank n, the least-squares solution of (4.1) is the unique vector $\underline{x}$ in $\mathbb{R}^n$ for which

$$||\underline{b} - A\underline{x}||_2 = \min_{\underline{y}\in\mathbb{R}^n} ||\underline{b} - A\underline{y}||_2 \quad (\text{where } ||\underline{u}||_2^2 := \underline{u}^*\underline{u}) . \tag{4.2}$$

For excellent surveys containing extensive bibliographies for such geodetic problems, see Golub and Plemmons [6], and Plemmons [18].

An equivalent formulation of the above least-squares problem is the following: determine (unique) vectors $\underline{x} \in \mathbb{R}^m$ and $\underline{r} \in \mathbb{R}^m$ such that

$$\underline{r} + A\underline{x} = \underline{b}, \quad \text{and} \quad A^T\underline{r} = \underline{0} . \tag{4.3}$$

Since A has full column rank n, we may assume, without loss of generality, that the rows of A have been permuted so that A has the block-partitioned form

$$A = \begin{bmatrix} A_1 \\ \\ A_2 \end{bmatrix} , \tag{4.4}$$

where A_1, an element of $\mathbb{R}^{n,n}$, is <u>nonsingular</u>. Partitioning the m-vectors $\underline{r}$ and $\underline{b}$ conformally with respect to the partitioning of A in (4.4), i.e.,

$$\underline{r} = \begin{bmatrix} \underline{v} \\ \\ \underline{w} \end{bmatrix} \quad \text{and} \quad \underline{b} = \begin{bmatrix} \underline{b}_1 \\ \\ \underline{b}_2 \end{bmatrix} , \text{ where } \underline{v}, \underline{b}_1 \in \mathbb{R}^n ; \; \underline{w}, \underline{b}_2 \in \mathbb{R}^{m-n} , \tag{4.5}$$

the system of equations (4.3) can be equivalently expressed as the following system of m+n linear equations in m+n unknowns:

$$C\underline{z} = \underline{d} , \tag{4.6}$$

where

$$C := \begin{bmatrix} A_1 & 0 & I \\ A_2 & I & 0 \\ 0 & A_2^T & A_1^T \end{bmatrix} ; \quad \underline{z} := \begin{bmatrix} \underline{x} \\ \underline{w} \\ \underline{v} \end{bmatrix} ; \tag{4.7}$$

$$\underline{d} := \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \underline{0} \end{bmatrix} .$$

Because A_1 is nonsingular, the form of the matrix C in (4.7) shows that C is also nonsingular, from which it follows that the vector $\underline{z}$ in (4.6) is uniquely determined.

Our interest in the reformulation (4.6) of the least-squares problem (4.2) stems from the fact that the block-SOR iterative method has been recently suggested as a practical means of solving (4.6). The purpose in this section is to review the new theoretical results concerning such block-SOR applications.

To define this iterative method, set D:= diag(C) = diag (A_1, I, A_1^T), so that D is a nonsingular block-diagonal matrix. The associated block-Jacobi matrix J for the matrix C of (4.7) is then given, as before, by

$$J := I - D^{-1}C = \begin{bmatrix} 0 & 0 & -A_1^{-1} \\ -A_2 & 0 & 0 \\ 0 & -A_1^{-T}A_2^T & 0 \end{bmatrix} =: \begin{bmatrix} 0 & 0 & B_1 \\ B_2 & 0 & 0 \\ 0 & B_3 & 0 \end{bmatrix} . \tag{4.8}$$

Next, on writing the block-Jacobi matrix J of (4.8) as the sum J = L + U where

$$L := \begin{bmatrix} 0 & 0 & 0 \\ B_2 & 0 & 0 \\ 0 & B_3 & 0 \end{bmatrix} ; \quad U := \begin{bmatrix} 0 & 0 & B_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \tag{4.9}$$

then the block-SOR iterative method, applied to (4.6), can be expressed as

$$\underline{z}^{(m+1)} = \mathcal{L}_\omega \underline{z}^{(m)} + (I - \omega L)^{-1} D^{-1} \underline{d}, \quad (m = 0, 1, \ldots), \tag{4.10}$$

where the block-SOR iterative matrix $\mathcal{L}_\omega$ is defined as usual by

$$\mathcal{L}_\omega := (I - \omega L)^{-1}\{(1-\omega)I + \omega U\}. \tag{4.11}$$

For the convergence properties of the block-SOR iterative method (4.10), it is essential to observe, as in Chen [3] and

Plemmons [18], that the block-Jacobi matrix J of (4.8) is a *consistently ordered weakly-cyclic of index* 3 *matrix* (cf. [22],[27]). Moreover, from (4.8), it directly follows that

$$J^3 = \text{diag}(-A_1^{-1}P^TPA_1;-PP^T;-P^TP) \quad (\text{where } P:= A_2A_1^{-1})$$
$$= \text{diag}(B_1B_3B_2;\ B_2B_1B_3;\ B_3B_2B_1) \ , \tag{4.12}$$

so that J^3 is similar to a real symmetric *negative semidefinite* matrix. As such, the eigenvalues of J^3 lie in the real interval

$$I_- := [-\rho^3(J),0] \ , \tag{4.13}$$

where $\rho(J)$ denotes the spectral radius of J. Because J is a consistently ordered weakly-cyclic of index 3 matrix, it is known (cf. Varga [22, Theorem 4.3]; [20, Thm. 4]) that the functional equation, coupling the eigenvalues λ of L_ω of (4.11) with the eigenvalues μ of J of (4.8), is given by

$$(\lambda + \omega - 1)^3 = \lambda^2\omega^3\mu^3 \ . \tag{4.14}$$

Using (4.14), the following precise domains for convergence and divergence of the block-SOR iterative, applied to (4.6), have been determined.

Theorem 4.1 ([15]). *The block-SOR iterative method of* (4.10), *applied to the least-squares matrix equation of* (4.6), *converges for*

$$0 < \omega < \omega_1(\beta) := \frac{2}{1+\beta},\ \text{when } 0 \leq \beta \leq 2,\ (\beta := \rho(J)), \tag{4.15}$$

converges for

$$\omega_2(\beta) := \frac{\beta-2}{\beta-1} < \omega < \omega_1(\beta),\ \text{when } 2 \leq \beta < 3 \ , \tag{4.16}$$

and diverges for all other real values of ω. *The optimal relaxation factor* $\omega_b = \omega_b(\beta)$ *is the unique positive root of*

$$4\beta^3\omega^3 + 27\omega - 27\omega = 0 \quad (0 \leq \beta < 3) \ , \tag{4.17}$$

and ω_b *satisfies*

$$\frac{1}{2} < \omega_b \leq 1 \quad \text{for all} \quad 0 \leq \beta < 3 \ . \tag{4.18}$$

Further, there holds

$$\rho(L_{\omega_b}) = 2(1-\omega_b) \quad \text{for all} \quad 0 \leq \beta < 3 . \tag{4.19}$$

It is clear from Theorem 4.1 that one can find values of ω for which the block-SOR iteration matrix L_ω is convergent, even when the block-Jacobi matrix J is divergent; for example, for each $\beta = \rho(J)$ satisfying $1 \leq \beta < 3$, there are intervals in ω (cf. (4.15) and (4.16)) for which $\rho(L_\omega) < 1$. In this respect, Theorem 4.1 extends what is known theoretically in the literature for such block-SOR applications. More important, however, is the fact that Theorem 4.1 greatly increases the applicability of the block-SOR iterative method to least-squares problems which arise, as was previously mentioned, in geodetical network problems.

It is also important to note that Theorem 4.1 corrects results in the literature for such least-squares applications. Under the assumptions of Theorem 4.1, one finds in Plemmons [18, p. 166] the statement that, for any $\rho(J) < 1$, the associated block-SOR iterative method converges for all ω satisfying

$$0 < \omega < \frac{3}{2} \quad (0 \leq \rho(J) < 1) , \tag{4.20}$$

whereas from (4.15) of Theorem 4.1, the correct statement is that the block-SOR iterative method converges in the subset of (4.20) consisting of all ω satisfying

$$0 < \omega < \frac{2}{1 + \rho(J)} \quad (0 \leq \rho(J) < 1) . \tag{4.21}$$

The same error occurs in Berman and Plemmons [2, p. 179].

As a counterpart of Theorem 4.1, the following new results on the convergence of the block-SOR iterative method have been determined when the eigenvalues of J^3 are nonnegative, i.e., the eigenvalues of J^3 lie in the interval

$$I_+ := [0, \rho^3(J)] . \tag{4.22}$$

Theorem 4.2 ([15]). *Let the block-Jacobi matrix* J *be a consistently ordered weakly-cyclic of index* 3 *matrix, such that the eigenvalues of* J^3 *are real and nonnegative. Then, the associated block-SOR iterative method converges for*

$$0 < \omega < \omega_3(\beta) := \frac{\beta+2}{\beta+1}\ , \quad \underline{\text{when}} \quad 0 \leq \beta < 1 \ (\beta := \rho(J))\ , \tag{4.23}$$

and diverges for all other real values of ω. The optimal relaxation factor $\omega_b = \omega_b(\beta)$ is the smallest positive root of

$$4\beta^3\omega^3 - 27\omega + 27 = 0 \quad (0 \leq \beta < 1)\ , \tag{4.24}$$

and ω_b satisfies

$$1 \leq \omega_b < \frac{3}{2} \quad \underline{\text{for all}} \quad 0 \leq \beta < 1\ . \tag{4.25}$$

Further, there holds

$$\rho(L_{\omega_b}) = 2(\omega_b - 1)\ . \tag{4.26}$$

Interestingly enough, while the proof of (4.24)-(4.26) is given in Varga [20] as the special case p = 3, the precise upper bound $\omega_3(\beta)$ of (4.23) for convergence is new. Previously, it had been shown in [20], under the assumptions of Theorem 4.2, that convergence of the associated block-SOR method holds in the subset of (4.23) defined by

$$0 < \omega < \frac{3}{2} \quad (0 \leq \beta := \rho(J) < 1)\ . \tag{4.27}$$

The results of Theorems 4.1 and 4.2 are graphically given in Figure 2. For convenience, we have introduced the variable $\tilde{\beta}$, where $\tilde{\beta} := \rho(J)$ if the eigenvalues of J^3 are nonnegative (cf. Theorem 4.2), and where $\tilde{\beta} = -\rho(J)$ if the eigenvalues of J^3 are nonpositive (cf. Theorem 4.1). Thus, an open bounded region Ω in the $\tilde{\beta}$-ω plane is obtained such that for each point of Ω (shown as the shaded region in Figure 2), the associated block-SOR matrix with relaxation factor ω (when applied to a system for which $\rho(J) = \tilde{\beta}$ or $\rho(J) = -\tilde{\beta}$) is convergent, and is divergent for all points in the complement of Ω. Also included in Figure 2 is the set of all optimum relaxation factors $\omega_b(\tilde{\beta})$, as a function of $\tilde{\beta}$; this appears as the dotted line in Figure 2.

V. REGULAR SPLITTINGS OF MATRICES

The theory of regular splittings of matrices has been a useful algebraic tool in the analysis of iterative methods for solving large systems of linear equations (cf. [2],[22], and

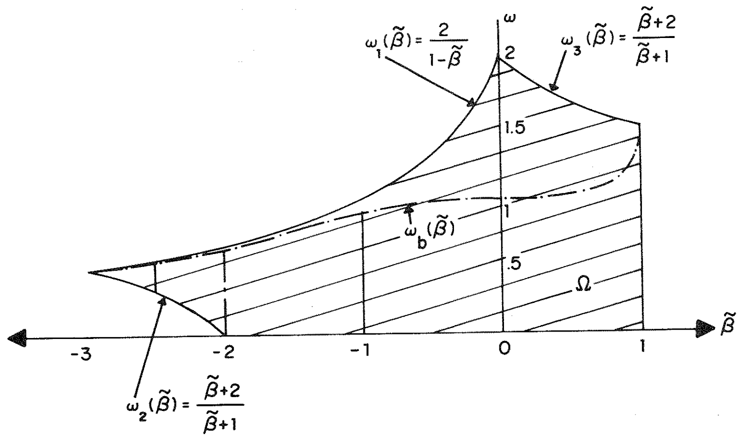

Figure 2

[27]). Our purpose in this section is to give some new comparison theorems for regular splittings of matrices, which generalize both the original results of Varga [21], and the subsequent unpublished thesis results of Woźnicki [25].

To begin, if $A \in \mathbb{C}^{n,n}$, then $A = M - N$ is said to be a regular splitting of A (cf. [21],[22]) if M is nonsingular, and if M^{-1} and N have all their entries nonnegative (written $M^{-1} \geq 0$ and $N \geq 0$). (We write $C \geq D$ and $C > D$ if $C - D \geq 0$ and if $C - D > 0$, respectively.) The following results of Theorem 5.1 and 5.2 are well-known.

Theorem 5.1 ([21]). *Let* $A = M - N$ *be a regular splitting of* A. *If* $A^{-1} \geq 0$, *then*

$$\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1 + \rho(A^{-1}N)} < 1. \tag{5.1}$$

Conversely, if $\rho(M^{-1}N) < 1$, *then* $A^{-1} \geq 0$.

Theorem 5.2 ([21]). *Let* $A = M_1 - N_1 = M_2 - N_2$ *be two regular splittings of* A, *where* $A^{-1} \geq 0$. *If* $N_2 \geq N_1$, *then*

$$\rho(M_2^{-1}N_2) \geq \rho(M_1^{-1}N_1) . \tag{5.2}$$

In particular, if $N_2 \geq N_1$ with $N_2 \neq N_1$, *and if* $A^{-1} > 0$, then

$$\rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1) \quad . \tag{5.3}$$

Suppose we are given the system of n linear equations in n unknowns:

$$A\underline{x} = \underline{k} \; , \tag{5.4}$$

where $A \in \mathbb{C}^{n,n}$ is a given matrix, and where $\underline{x}$ and $\underline{k}$ are in $\mathbb{C}^n$, with $\underline{k}$ being given. If $A = M_1 - N_1$ is a regular splitting of A, we can write (5.4) as

$$\underline{x} = M_1^{-1}N_1\underline{x} + M_1^{-1}\underline{k} \; , \tag{5.5}$$

which induces the following iterative method:

$$\underline{x}^{(m+1)} = M_1^{-1}N_1\underline{x}^{(m)} + M_1^{-1}\underline{k}, \quad (m = 0,1,\ldots) \; , \tag{5.6}$$

where $\underline{x}^{(0)}$ is an arbitrary vector in $\mathbb{C}^n$. When $A^{-1} \geq 0$, the iterative method of (5.6) is necessarily <u>convergent</u>, by Theorem 5.1, to the unique solution to (5.4), independent of the choice of $\underline{x}^{(0)}$. If $A = M_2 - N_2$ is another regular splitting of A, one similarly obtains the convergent iterative method

$$\underline{x}^{(m+1)} = M_2^{-1}N_2\underline{x}^{(m)} + M_2^{-1}\underline{k} \; , \quad (m = 0,1,\ldots) \; . \tag{5.7}$$

If $N_2 \geq N_1$ with $N_2 \neq N_1$ and if $A^{-1} > 0$, the iterative method of (5.6) is, from Theorem 5.2, <u>asymptotically faster</u> than the iterative method of (5.7).

From a practical point of view, it is of interest to know other techniques for determining which of two regular splittings of a matrix induces the asymptotically faster iterative method. One such technique, which is less well-known but still very useful, is the following thesis result of Woźnicki [25].

<u>Theorem 5.3</u> ([25]). <u>Let</u> $A = M_1 - N_1 = M_2 - N_2$ <u>be two regular splittings of</u> A, <u>where</u> $A^{-1} \geq 0$. <u>If</u> $M_1^{-1} \geq M_2^{-1}$, <u>then</u>

$$\rho(M_2^{-1}N_2) \geq \rho(M_1^{-1}N_1) \quad . \tag{5.8}$$

<u>In particular, if</u> $M_1^{-1} > M_2^{-1}$ <u>and if</u> $A^{-1} > 0$, <u>then</u>

$$\rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1) \quad . \tag{5.9}$$

As we shall see (cf. i) of Proposition 5.4), Woźnicki's Theorem 5.3 is more general than Theorem 5.2, and it is natural to ask if similar, more general results, based on the comparison of two regular splittings, can be deduced. To motivate our subsequent discussion, we state the result of

Proposition 5.4 ([4]). Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq 0$. Then,

i) $N_2 \geq N_1$ implies that $M_1^{-1} \geq M_2^{-1}$;

ii) $M_1^{-1} \geq M_2^{-1}$ implies that $A^{-1}N_2A^{-1} \geq A^{-1}N_1A^{-1}$;

iii) for each positive integer j, $A^{-1}N_2A^{-1} \geq A^{-1}N_1A^{-1}$ implies that $(A^{-1}N_2)^jA^{-1} \geq (A^{-1}N_1)^jA^{-1}$.

Obviously, Proposition 5.4 gives progressively weaker hypotheses. As our new generalization of the first parts of Theorems 5.2 and 5.3, we give

Theorem 5.5 ([4]). Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq 0$. Assume that there exists a positive integer j for which

$$(A^{-1}N_2)^jA^{-1} \geq (A^{-1}N_1)^jA^{-1} . \tag{5.10}$$

Then,

$$1 > \rho(M_2^{-1}N_2) \geq \rho(M_1^{-1}N_1) . \tag{5.11}$$

Continuing, let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq 0$. On setting

$$S := \{\text{positive integers } j: (A^{-1}N_2)^jA^{-1} \geq (A^{-1}N_1)^jA^{-1}\} , \tag{5.12}$$

we note that S not empty implies that Theorem 5.5 can be applied. It is easy to show that S is closed under addition, so that if 1 is in S, then S consists of all positive integers. Similarly, if S contains two relatively prime integers, then S necessarily contains all sufficiently large positive integers. Matrices A and two regular splittings can, however, be constructed (cf. [4]) such that the smallest positive integer in the associated set S can be made to be any preassigned positive integer!

For results which provide partial converses to Theorem 5.5, as suggested by the second parts of Theorems 5.2 and 5.3, we state the new result of

Theorem 5.6 ([4]). *Let* $A = M_1 - N_1 = M_2 - N_2$ *be two regular splittings of* A, where $A^{-1} > 0$. *If*

$$\rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1) \quad , \tag{5.13}$$

there exists a positive integer j_o *for which*

$$(A^{-1}N_2)^j A^{-1} > (A^{-1}N_1)^j A^{-1}, \quad \text{for all} \quad j \geq j_o \ . \tag{5.14}$$

Thus, (5.13) *implies that the set* S *of* (5.12) *contains all sufficiently large positive integers.* *Conversely, if there is an integer* j *for which*

$$(A^{-1}N_2)^j A^{-1} > (A^{-1}N_1)^j A^{-1} \quad , \tag{5.15}$$

then (5.13) *is valid.*

We finally remark that the case j = 1 of (5.15) generalizes the second parts of *both* Theorems 5.2 and 5.3, as shown in [4].

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